Accurate solution of light propagation in optical waveguides using Richardson extrapolation

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Abstract

Light propagation in optical waveguides is studied in the paraxial approximation using Richardson extrapolation and the mid-step Euler finite difference algorithm. Highly accurate solutions can be efficiently obtained using this combined approach. Since discretization errors in both the transverse and propagation dimensions are greatly reduced, accuracies on the order of $10^{-11}$ or better are obtained. Richardson extrapolation allows us to use transparent boundary conditions, which normally can only be used with the Crank–Nicholson method. Richardson extrapolation also allows us to stabilize the mid-step Euler method which is explicit and thus use it in place of Crank–Nicholson method which is implicit. Implicit schemes do not vectorize well on the CRAY machines which we are using while explicit schemes do. Consequently, the approach presented here is competitive in CPU cost to the Crank–Nicholson method while generating results of significantly larger accuracy. To illustrate this approach we apply it to the study of a straight, integrated optical waveguide and a y-junction and compare the results to the results from a Crank–Nicholson approach.

1. Introduction

The development of all-optical communication systems and many other optical systems relies heavily on the progress of research in optical integrated circuits. In optical integrated circuits, one typically uses guided waves to perform tasks such as switching, multiplexing, modulation, demodulation, and analog-to-digital conversion. Accurate analysis of such guided wave devices will greatly aid the design and optimization process. In general, numerical methods are used to investigate the propagation characteristics of guided wave devices. The beam propagation method based on the fast Fourier transform is currently the most widely used approach to study complex photonic structures [1–3]; scalar and vector finite difference schemes which are based on direct discretization of the scalar wave equation have also been proposed [4–7]; more

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recently, a hybrid finite difference scheme using the fast Fourier transform has been implemented [8]. In choosing a numerical method, one wishes to ensure stability and to minimize the CPU cost for a given, desired accuracy. For a given algorithm, a straightforward way to improve accuracy is to increase the density of the transverse grid and decrease the propagation step size, however, the computational cost quickly becomes prohibitive for the second order schemes which are most often used [1–8]. Higher order schemes can achieve higher accuracies, but they are typically cumbersome to implement. Richardson extrapolation overcomes this latter disadvantage.

It is important to develop highly accurate, yet rapid schemes for solving the paraxial wave equation in order to be able to separate inaccuracies inherent in the numerical methods from inaccuracies due to the paraxial wave approximation itself and to provide a baseline against which the accuracy of standard schemes can be measured in situations, where analytic solutions do not exist — for example, situations in which continuous radiation is produced. In this paper, we use an algorithm based on Richardson extrapolation to efficiently and simply obtain a highly accurate solution of the paraxial wave equation. In any numerical scheme, a differential equation is replaced by a finite difference equation which is then iterated to obtain a solution. In general, if the discretization step sizes are sufficiently small, the solution of the finite difference equation approximates that of the differential equation. The two solutions converge when the step size equals zero. In Richardson extrapolation, numerical solutions with lesser accuracy are extrapolated to zero step size in order to obtain a more accurate solution of the difference equation [9–11]. This method is commonly used in numerical analysis [12,13] and has also been applied in disciplines such as acoustics [14] and nuclear engineering [15]; however, this paper presents its first application to optical waveguide problems.

In the following, we will discuss Richardson extrapolation and its implementation to solve the paraxial wave equation with one transverse dimension based on the explicit, mid-step Euler finite difference method. We stress that Richardson extrapolation is a simple algebraic procedure that can be used in conjunction with any numerical method to improve its accuracy. The explicit mid-step Euler method vectorizes well on a CRAY-YMP, the computer that we used in the investigations reported here. By contrast, implicit schemes involve tridiagonal matrix inversions and do not vectorize well. Thus, the explicit mid-step Euler method is significantly more efficient, but it is unconditionally unstable. Consequently, it has been rarely used in problems with one transverse dimension. However, it has long been known that Richardson extrapolation can stabilize unstable schemes [16], and we have found that it stabilizes the explicit mid-step Euler method in optical waveguide problems. As a consequence, the CPU cost of our approach is competitive with implicit schemes which have far lower accuracy. This opens up the possibility of building efficient three-dimensional codes.

As a particular example, we study the evolution of a Gaussian input pulse along a straight, single-mode, step-index waveguide. We use transparent boundary conditions [17,18]. These boundary conditions are meant to be used in conjunction with the implicit Crank–Nicholson method, but they work well with our explicit method when combined with Richardson extrapolation for reasons which are stated in the main text. The purpose of this example is to examine the evolution of an initial pulse which is not perfectly matched to the waveguide and produces radiation — a case which often occurs in practice. We show that by using Richardson extrapolation first in the propagation direction and then, at the end of the simulation, in the
transverse direction, that is possible to not only eliminate much of the error due to a finite step size, but also much of the transverse discretization error. We also study a Y-junction optical waveguide excited with a Gaussian input pulse and show that this numerical approach is applicable to more complex optical devices.

2. Computational approach

2.1. Richardson extrapolation

In Richardson extrapolation, one uses a sequence of estimates, that are obtained by varying the step size of a numerical calculation, to extrapolate to zero step size [12]. In the mid-step Euler method, we may write the solution obtained after propagating for a step as

\[ A(h, N) = S(h) + \varepsilon_2(h) \frac{1}{N^2} + \varepsilon_3(h) \frac{1}{N^3} + \cdots, \]

where \( A(h, N) \) is the outcome of a numerical calculation over a short but finite interval \( h \), \( S \) is the exact solution, and \( \Delta = h/N \) is the step size used inside the interval. More generally, one can have a term proportional to \( 1/N \) in the sequence; however, the mid-step Euler method which we employed in our calculations is second-order accurate. Hence, the lowest order error term after each sub-step propagation is proportional to \( 1/N^2 \), and, since there are \( N \) steps inside \( h \), the first error term will be proportional to \( 1/N^2 \). From Eq. (1), the leading error term can be eliminated by taking a linear combination of the results calculated using two different values of \( N \). For example, we find

\[ \frac{4A(h, 2) - A(h, 1)}{3} = S(h) - \frac{1}{6} \varepsilon_3(h) + \cdots \]

This procedure can be repeated to arbitrarily high order by generating a sequence of solutions with different values of \( N \). We use step sizes of \( h, h/2, h/4, h/8 \), etc., but any sequence of decreasing step sizes can be employed [12]. Schematically, the Richardson extrapolation can be represented as

\[
\begin{align*}
N = 1; & \quad \Delta = h \rightarrow A_{11} \\
N = 2; & \quad \Delta = h/2 \rightarrow A_{12} \\
N = 4; & \quad \Delta = h/4 \rightarrow A_{13} \\
N = 8; & \quad \Delta = h/8 \rightarrow A_{14}
\end{align*}
\]

\[
\begin{align*}
A_{21} & \quad A_{31} \\
A_{22} & \quad A_{32} \\
A_{23} & \quad A_{41}
\end{align*}
\]

where the \( A_{ij} \) are the successive solutions obtained by the extrapolation. The first index \( i \) represents the \( i \)th stage of Richardson extrapolation and the second index \( j \) corresponds to the number of substeps used in obtaining the solution. The extrapolated solution at the \( i \)th stage is \((i-1)\) orders more accurate than the original solution. Note that the \( A_{ij} \) when \( i > 1 \) are determined by simple algebraic calculations as demonstrated in Eq. (2). After each stage of extrapolation, the accuracy of the solution can be estimated by comparing the solution with the best solution obtained in the previous Richardson stage. One can apply Richardson extrapolation...
tion if the solution of the equation is smooth and does not contain any singular points inside the interval of integration.

For a particular numerical method and a discretized grid, the choice of the propagation step size and the desired accuracy limits determines the amount of extrapolation required. The propagation step size has to be chosen carefully. If it is too large, extrapolation has to be carried to too many stages. On the other hand, if the step size is too small, too many number of steps are required to propagate for a given length. There is an optimum size for each propagation step which depends on both the CPU time required to do the algebraic procedure involved in extrapolation and the CPU time required to do the basic propagation step. The optimum step size also depends on the solution and its derivatives. Ideally, a code should include a built-in algorithm to choose a variable step size for the propagation of the wave solution. A possible algorithm to choose a optimum step size is discussed in [12]. The application of variable step sizes to waveguide problems is discussed elsewhere.

2.2. Mid-step Euler, finite difference method

To model light wave propagation in optical waveguide devices, we use the paraxial wave equation [4],

\[-2 j k_0 n_0 \frac{\partial E}{\partial z} = \frac{\partial^2 E}{\partial x^2} + k_0^2 \left[ n^2(x, z) - n_0^2 \right] E, \tag{3}\]

where \(E\) is the electric field vector of a TE wave and \(k_0\) is the wavenumber in free space. The parameter \(n_0\) is the reference refractive index, and \(n(x, z)\) is the cross-section index profile. We solve Eq. (3) numerically using the mid-step Euler finite difference method, which is given by

\[-2 j k_0 n_0 \frac{E^{m+\frac{1}{2}} - E^{m}}{\Delta z/2} = \frac{E^{m+1} - 2E^m + E^{m-1}}{\Delta x^2} + k_0^2 \left[ \left( n^m \right)^2 - n_0^2 \right] E^m, \tag{4a}\]

\[-2 j k_0 n_0 \frac{E^{m+1} - E^{m}}{\Delta z} = \frac{E^{m+1} - 2E^{m+\frac{1}{2}} + E^{m-1}}{\Delta x^2} + k_0^2 \left[ \left( n^{m+\frac{1}{2}} \right)^2 - n_0^2 \right] E^{m+\frac{1}{2}}, \tag{4b}\]

where \(E^i = E(i\Delta x, j\Delta z)\), where \(\Delta x\) and \(\Delta z\) are step sizes in the transverse and propagation directions respectively.

The mid-step Euler method is an explicit scheme and easily vectorizable; hence, it requires less computer time per step on a vector computer, such as the CRAY Y-MP that we are using, than either methods that use Fourier transforms or implicit methods that require inversions of tridiagonal matrices. Despite the efficiency with which it can be solved, Eq. (4) is unconditionally unstable; i.e., any numerical error generated will grow exponentially. However, it has long been known that by using Richardson extrapolation and by choosing a sufficiently small step size, it is often possible to stabilize unstable schemes and obtain accurate solutions [16]. We show here that Richardson extrapolation stabilizes the explicit mid-step Euler method in the problems we are considering.
2.3. 2-D Extrapolation

When solving the paraxial equation using the mid-step Euler, finite difference equations, both the propagation and the transverse dimensions are discretized. To obtain an accurate solution, the discretization errors arising from both the transverse grid and the propagation step size have to be eliminated. We solve this problem by employing Richardson extrapolation in both directions. The solution obtained after propagating a step forward using the mid-step Euler, finite difference method can be written as

\[
A(h_z, h_x, N, M) = S(h_z, h_x) + f_0(h_z, h_x, M) + f_1(h_z, h_x, M) \frac{1}{N} + f_2(h_z, h_x, M) \frac{1}{N^2} + \cdots \tag{5}
\]

where \(S\) is the exact solution over a very short interval of \(h_z\) in the propagation direction and a width of \(h_x\) in the transverse dimension, \(\Delta z = h_z / N\) is the step size used to propagate inside \(h_z\), and \(\Delta x = h_x / M\) is the grid size used inside \(h_x\). The parameters \(M\) and \(N\) are positive integers, and

\[
f_i(h_z, h_x, M) = \eta_{0,i}(h_z, h_x) + \eta_{2,i}(h_z, h_x) \frac{1}{M^2} + \eta_{4,i}(h_z, h_x) \frac{1}{M^4} + \cdots \tag{6}
\]

The coefficients \(\eta_{0,0}\) and \(\eta_{0,1}\) are equal to zero as the method is second order accurate. Unlike Eq. (1), error terms now consist of terms involving \(N, M\), and products of \(N\) and \(M\) due to the 2-dimensional discretization.

Error terms in Eq. (5) that involve \(N\) and cross terms involving \(N\) and \(M\) can be eliminated by using different values of \(N\) with Richardson extrapolation as in the one-dimensional case described earlier, but the error terms which contain only \(M\) are not eliminated. By extrapolating on the solution with different values of \(M\), we can eliminate these terms as well.

To eliminate the discretization errors due to the finite grid size in the transverse dimension and the finite propagation step, we have to choose several different grid sizes and solve the propagation problem on all of them. Solutions with different grid sizes can then be extrapolated in the transverse dimension to reduce the remaining errors. The transverse extrapolation can only be carried out on the common points of the different grids and so the maximum size of the extrapolated vector corresponds to that of the coarsest grid. Recently, a scheme which combines interpolation and Richardson extrapolation has been described which obtains an accurate solution on the finest grid size used [19], but we do not apply this approach here. The accumulated transverse discretization error at the end of the propagation will be more than the set threshold used in each propagation step. The accumulation will depend upon the phase of the different errors added at each step along the propagation. In principle, growth of the error can be checked by periodically carrying out an extrapolation in the transverse dimension which eliminates the transverse discretization errors. In fact, it is not necessary to carry out the transverse extrapolation on every propagation step, and, indeed, for the example problems we are considering, it is only necessary to carry out the transverse extrapolation at the end of the entire propagation length to obtain high accuracy!
3. Boundary conditions

In the paraxial wave equation, the initial field $E(x, 0)$ is given at $z = 0$ as a function of $x$, and the evolution of the field in $z$ is computed from Eq. (3). In addition, it is necessary to specify boundary conditions at the limits of the computational window in $x$. A physically reasonable boundary condition is that any part of the field which reaches the boundary is radiated out, and its energy is lost from the system. This condition is somewhat tricky to implement as the boundary of the computational domain is finite, and at a finite boundary it is difficult to distinguish between the part of the field that is propagating transversely and the part that is propagating longitudinally.

Recently, Hadley [17,18] proposed transparent boundary conditions in which the transverse complex wave vector is estimated from the previous step by calculating the ratio of the field at the two grid points nearest to the edge of the computational window. Using this transverse complex wave vector, one can eliminate any incoming flux which forces the energy to only flow outward. Transparent boundary conditions are both more accurate and require less computer time to implement than does the traditional approach of adding an absorbing region to the computational window. In addition, the implementation of transparent boundary conditions is relatively independent of the waveguide structure simulated and is therefore more robust than conventional absorbers.

To date transparent boundary conditions have only been implemented with the implicit Crank–Nicholson scheme. The reason is that the Crank–Nicholson method has an exact flux invariant, in contrast to all other numerical schemes which have been used to date on optical waveguide problems. One can use this flux invariant to ensure that the energy flux is outgoing in the Crank–Nicholson method, but that is not possible in other methods and one finds empirically that this approach does not work well. However, the original paraxial wave equation also has an exact flux invariant. While the mid-step Euler method does not have an exact flux invariant, when we extrapolate to zero step size using Richardson extrapolation, the error in the flux also tends to zero. Thus, we can always ensure that the energy flow is outgoing. To demonstrate this result, we first write

$$
|E_{i+1}^n|^2 - |E_i^n|^2 = \frac{j\Delta z}{8k\Delta x^2} \left[ \left( E_i^n + E_i^{n+1} \right)^* \cdot \left( E_{i+1}^n + E_{i+1}^{n+1} \right) 
+ \left( E_i^n + E_i^{n-1} \right)^* \cdot \left( E_{i-1}^n + E_{i-1}^{n+1} \right) - c.c. \right] + G(\Delta z^3)
$$

(7)

in the mid-step Euler method, where $G(\Delta z^3)$ consists of a contribution which is $O(\Delta z^3)$. In the Crank–Nicholson method, $G(\Delta z^3) = 0$ which is why it is possible to apply transparent boundary conditions [17,18]. Eq. (7) can be written as

$$
|E_{i+1}^n|^2 - |E_i^n|^2 = \Delta z (F_{i-1(1/2)} - F_{i+1(1/2)}) + G(\Delta z^3),
$$

(8)

where $F$ is defined as

$$
F_{i+1(1/2)} = -\frac{i}{8k\Delta x^2} \left[ \left( E_i^n + E_i^{n+1} \right)^* \cdot \left( E_{i+1}^n + E_{i+1}^{n+1} \right) - \left( E_i^n + E_i^{n+1} \right) \cdot \left( E_{i+1}^n + E_{i+1}^{n+1} \right)^* \right].
$$

(9)
Hence, in a sub-step of the Richardson extrapolation of length $\Delta z = h/N$, we find

$$|E_i^{n+1}|^2 - |E_i^n|^2 = \frac{h}{N} (F_{i-1/2} - F_{i+1/2}) + g_1 \frac{h^3}{N^3} + g_2 \frac{h^4}{N^4} + \cdots,$$

where we have expanded $G(\Delta z^3)$ in a Taylor series. Hence, over the entire step of length $h$, if follows that

$$|E_i^{n+1}|^2 - |E_i^n|^2 = h(F_{i-1/2} - F_{i+1/2}) + \alpha_1 \frac{h^3}{N^2} + \alpha_2 \frac{h^4}{N^3} + \cdots,$$

where, once again, $\alpha_1, \alpha_2, \ldots$ indicate the coefficients of an appropriate Taylor series. When we carry out Richardson extrapolation, we are in effect extrapolating $N \rightarrow \infty$ and the deviation from exact flux conservation becomes negligibly small. The flux $F_{i+1/2}$ defined in Eq. 9 is identical to Hadley’s definition, and we may therefore use precisely the same condition which he used to keep the outgoing flux positive at both boundaries, i.e., the real part of the transverse wave vector $k_x$ should obey

$$0 < \text{Re}(k_x) < \frac{\pi}{\Delta x}.$$

4. Numerical results

We considered the problem of wave propagation in a straight, integrated optical waveguide and a Y-junction waveguide excited with a Gaussian input pulse. We calculated electric fields at each propagation step using Eq. (4) and then extrapolated the solutions until the error was below threshold. We used error thresholds of $10^{-8}$ and $10^{-12}$ to obtain two different sets of solutions with different accuracies. The error estimate is defined as $\epsilon = \sum_{i=1}^{n} |E_i - E'_i|^2 \Delta x$ where the subscript $i$ indicates the grid location in transverse dimension, $n$ is the number of grid points, $E$ is the extrapolated solution, and $E'$ is the solution obtained in the previous Richardson stage. The input pulse is normalized to unit power. The total propagation length is 4000 $\mu$m for the straight waveguide problem and 500 $\mu$m for the Y-junction problem. We carried out our calculations with 512, 1024, and 2048 grid points in the transverse dimension $x$.

![Fig. 1. Straight waveguide problem geometry; $n_g$ and $n_s$ are the refractive indices of the guiding and the substrate regions respectively.](image-url)
At the end of the simulations, the solutions obtained for different numbers of grid points were extrapolated in $x$ to reduce errors due to transverse discretization.

Figs. 1 and 2 describe the problem geometry for the straight waveguide and Y-junction problems. For both problems, the width of the guiding region is 4 $\mu$m and the computational window is 60 $\mu$m. The branching angle of the Y-junction is 3$^\circ$. The refractive indices of the guiding and surrounding regions are 3.38 and 3.377 respectively. The reference refractive index $n_0$ is 3.377. The wavelength is 1.15 $\mu$m and the FWHM of the input Gaussian pulse is 4.828 $\mu$m. In the straight waveguide problem, we used initial propagation step sizes of 0.4, 0.067, and 0.013 $\mu$m for, respectively, 512, 1024, and 2048 transverse grid points when we set the error threshold at $10^{-8}$, while we used step sizes of 0.2, 0.067, and 0.013 respectively when we set the error threshold at $10^{-12}$. For the Y-junction problem, we used initial step sizes of 0.15, 0.06, and 0.015 with the $10^{-8}$ error threshold and 0.05, 0.012, and 0.004 with the $10^{-12}$ error threshold.

In Fig. 3, we show the error estimate for each $z$-step for up to 3 stages of Richardson extrapolation for the wave evolution in the straight waveguide. In the case shown here, the error threshold is $10^{-12}$, the basic step size is 0.25 $\mu$m, and there are 512 grid points. We use
only as many stages as are required for the error to go below threshold. A dramatic reduction in error is obtained with each additional iteration. The graph shows the simulation for a distance of 250 \( \mu \text{m} \).

In Fig. 4, we plot the radiation intensity at distances of 1000, 2000, 3000 and 4000 \( \mu \text{m} \) along the straight waveguide. We use a semilog plot which shows the radiation leaving the guiding region. Figures 5 and 6 show the pulse evolution as the pulse propagates along the straight waveguide and the Y-junction waveguide. The solutions obtained with error thresholds of \( 10^{-8} \) and \( 10^{-12} \) are compared and the difference is taken as an estimate of the actual error in the solution with an error threshold of \( 10^{-8} \). This error is depicted in Fig. 7 for the straight waveguide. The error grows along the propagation length because of the accumulation at each propagating step. The accumulation depends on the sign and the phase of the error arising at each step. After 1000 \( \mu \text{m} \), most of the radiation has left the neighborhood of the waveguide, and the error accumulation is close to exponential because its phase is nearly fixed. The error accumulation in the Y-junction problem is shown in Fig. 8. Its character differs significantly from a straight waveguide. There is a rapid, almost instantaneous increase in the error at 100
Fig. 6. The evolution of the Gaussian pulse through the Y-junction waveguide.

Fig. 7. Error in the propagating solution evaluated with a $10^{-8}$ error threshold while propagating in $z$ for a straight waveguide.

Fig. 8. Error in the propagating solution evaluated with a $10^{-8}$ error threshold while propagating in $z$ for a Y-junction waveguide.
Table 1
Estimated error in the straight waveguide solution after two stages of Richardson extrapolation in the transverse dimension. Error thresholds of $10^{-8}$ and $10^{-12}$ were used for the z-propagation

<table>
<thead>
<tr>
<th>Distance $\mu m$</th>
<th>$10^{-8}$</th>
<th>$10^{-12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>$1.7 \times 10^{-11}$</td>
<td>$9.7 \times 10^{-12}$</td>
</tr>
<tr>
<td>2000</td>
<td>$2.7 \times 10^{-10}$</td>
<td>$1.6 \times 10^{-11}$</td>
</tr>
<tr>
<td>3000</td>
<td>$6.8 \times 10^{-9}$</td>
<td>$1.2 \times 10^{-11}$</td>
</tr>
<tr>
<td>4000</td>
<td>$6.3 \times 10^{-8}$</td>
<td>$1.1 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

$\mu m$ which is the branching point. Beyond 250 $\mu m$, there is almost no increase in the error because the error phase is continuously changing due to the branching.

After the transverse extrapolation, the error is of the order of $10^{-8}$ when the error threshold is $10^{-8}$ in both the straight and Y-junction problems. The estimated error after carrying out two stages of transverse Richardson extrapolation at several distances is shown in Table 1 and Table 2 for the straight waveguide and the Y-junction respectively. In column 1 we used an error threshold of $10^{-8}$ while propagating in the z direction, and in column 2 we used an error threshold of $10^{-12}$. Note that in the straight waveguide when the error threshold is set at $10^{-8}$ the estimated accuracies are actually lower than the error threshold for 1000 $\mu m$–3000 $\mu m$. That happened because the actual error was significantly below threshold in this case as determined by comparison with the case in which the error threshold was $10^{-12}$. This comparison yields an error which is of the same order of magnitude as the estimate given by the transverse extrapolation. The accuracy of the solutions are improved several orders of magnitude by transverse extrapolation because the errors in the solution with $n = 512$, $n = 1024$, and $n = 2048$ grid points are strongly correlated. The correlation between the errors in the solutions with different $n$ are measured. The correlation coefficients are found to be in the range of 0.5 to 0.8 for both the $10^{-8}$ and $10^{-12}$ error thresholds in the straight waveguide problem. The correlation coefficients are in the range of 0.4 to 0.6 in the Y-junction problem. The Y-junction transverse extrapolation is only able to give an accuracy of $10^{-8}$ even in the case where an error threshold of $10^{-12}$ is used while propagating in the z direction. This might be due to the large error accumulation at the branching point of the Y-junction which reduces the correlation between the error phases at different $n$.

Table 2
Estimated error in the Y-junction solution after two stages of Richardson extrapolation in the transverse dimension. Error thresholds of $10^{-8}$ and $10^{-12}$ were used for the z-propagation

<table>
<thead>
<tr>
<th>Distance $\mu m$</th>
<th>$10^{-8}$</th>
<th>$10^{-12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>125</td>
<td>$1.9 \times 10^{-8}$</td>
<td>$7.8 \times 10^{-9}$</td>
</tr>
<tr>
<td>250</td>
<td>$2.7 \times 10^{-8}$</td>
<td>$3.8 \times 10^{-8}$</td>
</tr>
<tr>
<td>375</td>
<td>$6.4 \times 10^{-8}$</td>
<td>$3.7 \times 10^{-8}$</td>
</tr>
<tr>
<td>500</td>
<td>$6.9 \times 10^{-8}$</td>
<td>$4.0 \times 10^{-8}$</td>
</tr>
</tbody>
</table>
Finally, we compare the efficiency of the proposed scheme with the Crank–Nicholson implicit finite difference method. For this comparison, we use 2048 transverse grid points. In the straight waveguide problem, using a step size of 0.0285 μm, we find that the Crank–Nicholson method takes approximately 18 minutes of CRAY-YMP time and attains an accuracy on the order of $10^{-6}$. The accuracy is determined by comparison with the highly accurate solution obtained by setting the error threshold at $10^{-12}$. Using the mid-step Euler method and Richardson extrapolation, only 14 minutes is required to obtain an accuracy on the order of $10^{-11}$. For a Y-junction waveguide, when using the mid-step Euler method with Richardson extrapolation, 2 minutes of Cray time is required to obtain an accuracy of $10^{-8}$, while the Crank–Nicholson method with a step size of 0.02 μm required 4 minutes to obtain an accuracy of $10^{-6}$. These results demonstrate that Richardson extrapolation is an efficient way to obtain accurate solutions for optical waveguide problems of practical interest.

5. Conclusions

In this paper, we present a simple and efficient approach, based on a combination of the mid-step Euler method with Richardson extrapolation, to obtain an accurate solution of the paraxial wave equation. The extrapolation scheme is used to reduce discretization errors in both the transverse and propagation directions. We apply this approach to both a straight waveguide and a Y-junction waveguide. Solutions with accuracies on the order of $10^{-8}$ to $10^{-12}$ are obtained. Comparison with the Crank–Nicholson method shows that this approach is efficient as well as highly accurate. The applicability of transparent boundary conditions, and vectorizability makes this explicit scheme a strong candidate for future three-dimensional code development.

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