4. Sampling and Aliasing
Sampling

• Most real signals are continuous-time (analogue) signals
  • *E.g. speech, audio, etc.*
• Computers have much difficulty in handling continuous-time signals
• Need sampling
  ⇒ Extract samples of the signal at some particular time instants
Continuous-to-Discrete or Analogue-to-Digital

$x(t)$

$x[n] = x(nT_s)$

$T_s = 1/f_s$

What is the value of $T_s$?

Normal CD music
$f_s = 44.1$ kHz

Sampled at
$f_s = 16$ kHz

Sampled at
$f_s = 8$ kHz
Sampling Sinusoids

Continuous Waveform: \( x(t) = \cos(2\pi 100t) \)

Sampled Signal: \( x[n] = x(nT_s) = \cos(2\pi 100nT_s) \), with \( T_s = 0.0005 \)

Sampled Signal: \( x[n] = x(nT_s) = \cos(2\pi 100nT_s) \), with \( T_s = 0.002 \)

\( f = 100\text{Hz} \)

\( f_s = 2000\text{Hz} \)

\( f = 100\text{Hz} \)

\( f_s = 500\text{Hz} \)
$x(t) = A \cos(2\pi ft + \phi)$

$x[n] = x(nT_s) = A \cos(2\pi fnT_s + \phi)$

$$= A \cos \left( 2\pi n \frac{f}{f_s} + \phi \right)$$

$$= A \cos(\hat{\omega}n + \phi)$$

where $\hat{\omega} = 2\pi \frac{f}{f_s}$ is the so-called discrete-time radian frequency
Spectrum of Sampled Sinusoids

• Assume $f = 100\text{Hz}$, $f_s = 300\text{Hz}$, $A = 1$ and $\phi = 0$

\[
x(t) = \cos(2\pi 100t) \quad \text{Original sinusoid}
\]

\[
x[n] = \cos(2\pi 100n / 300) \quad \text{Sampled sinusoid}
\]

• From Fourier series, it is known that the spectrum of $x(t)$, i.e. $X_k$ is as follows:

\[
\begin{array}{c|c|c}
-100 & 0 & 100 \\
\hline
0 & 1/2 & 1/2
\end{array}
\]

Spectrum of original sinusoid
4. Sampling and Aliasing

• From Fourier series, we know that

\[ X_k = \frac{2}{T_0} \int_0^{T_0} x(t)e^{-j2\pi kt/T_0} \, dt \]

• If \( x(t) \) is sampled to \( x(nT_s) \)

\[ X_k^P = \frac{2T_s}{T_0} \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi knT_s/T_0} \]

\[ = \frac{2}{N} \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi kn/N} \]
4. Sampling and Aliasing

$N$ is the number of samples in one period

Continuous Waveform: $x(t) = \cos(2\pi 100t)$

Sampled Signal: $x[n] = x(nT_s) = \cos(2\pi 100nT_s)$, with $T_s = 0.0005$

$N = \frac{T_0}{T_s} = 20$

Sampled Signal: $x[n] = x(nT_s) = \cos(2\pi 100nT_s)$, with $T_s = 0.002$

$N = \frac{T_0}{T_s} = 5$
\[ X_k^P = \frac{2}{3} \sum_{n=0}^{2} \cos\left(\frac{2\pi n}{3}\right)e^{-j2\pi kn/3} \]

\[ = \frac{1}{3} \sum_{n=0}^{2} \left( e^{j2\pi n/3} + e^{-j2\pi n/3} \right)e^{-j2\pi kn/3} \]

\[ = \frac{1}{3} \sum_{n=0}^{2} \left( e^{j2\pi (1-k)n/3} + e^{-j2\pi (1+k)n/3} \right) \]
4. Sampling and Aliasing

Let’s consider a particular \( k \), e.g. \( k = 0 \)

\[
\sum_{n=0}^{2} e^{j2\pi n/3} = 1 + e^{j2\pi /3} + e^{j2\pi 2/3}
\]

\[
= 1 + \cos(2\pi /3) + j\sin(2\pi /3) \\
+ \cos(2\pi 2 /3) + j\sin(2\pi 2/3)
\]

\[
= 1 - 0.5 + j0.866 - 0.5 - j0.866
\]

\[
= 0
\]

In fact

\[
\sum_{n=0}^{N-1} e^{j2\pi mn/N} = 0 \quad \text{if} \quad e^{j2\pi mn/N} \neq 0
\]
4. Sampling and Aliasing

- In general

\[
\sum_{n=0}^{N-1} e^{j2\pi mn/N} = \begin{cases} 
0 & \text{otherwise} \\
N & \text{for } m = \text{multiple of } N
\end{cases}
\]

since

\[
e^{j2\pi (0)n/N} = e^{j2\pi (N)n/N} = e^{j2\pi (2N)n/N} = e^{j2\pi (-3N)n/N} = 0
\]
4. Sampling and Aliasing

\[
X_k^P = \frac{1}{3} \sum_{n=0}^{2} \left( e^{j2\pi(1-k)n/3} + e^{-j2\pi(1+k)n/3} \right)
\]

\[
= \begin{cases} 
1 & \text{if } (1-k) = \text{multiple of 3} \quad \text{or} \\
0 & \text{otherwise} \\
\end{cases}
\]

Magnitude Spectrum for sampled sinusoid
Spectrum Analysis and Filtering

4. Sampling and Aliasing

Magnitude Spectrum for sampled sinusoid

Spectrum of sampled sinusoid

Magnitude Spectrum for original sinusoid

Spectrum of original sinusoid

Ideal low pass filter
A/D and D/A conversions

A/D converter

\[ f_s = \frac{1}{T_s} \]

D/A converter

Sampler

Ideal Low Pass Filter
• Very often the discrete-time signal spectrum is expressed using discrete-time radian frequency

\[ \hat{\omega} = 2\pi \frac{kf_0}{f_s} \]
4. Sampling and Aliasing

**Aliasing**

- Assume \( f = 100\text{Hz}, f_s = 200\text{Hz}, A = 1 \) and \( \phi = 0 \)

\[
\hat{\omega} = 2\pi \frac{k f_0}{f_s}
\]

**Magnitude Spectrum for sampled sinusoid**

**Ideal low pass filter**
4. Sampling and Aliasing

• Assume $f = 100$Hz, $f_s = 100$Hz, $A = 1$ and $\phi = 0$

$$\hat{\omega} = 2\pi \frac{kf_0}{f_s}$$

Only get a DC when an ideal low-pass filter is used
4. Sampling and Aliasing

Samples with $f_s = 100\text{Hz}$

100 Hz cosine wave $f_0 = 100\text{Hz}$

Just a DC
Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\text{max}}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$ if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\text{max}}$.
Discrete-to-Continuous Conversion

• Achieve by low pass filtering
  ⇒ Smooth out the sharp changes in the signal as much as possible

The simplest low pass filter is a capacitor, which works like a reservoir to store the voltage of the samples
• Let $p(t) = 1$ rectangular pulse

\[ x[n] \]

\[ t \]

0 $T_s$

• Such low pass filter operation can be mathematically expressed as

\[ x(t) = x[0]p(t) + x[1]p(t - T_s) + x[2]p(t - 2T_s) + x[3]p(t - 3T_s) \]
• In general, if we have \( N \) samples,

\[
x(t) = x[0]p(t) + x[1]p(t - T_s) + x[2]p(t - 2T_s) \\
+ \ldots + x[N - 2]p(t - (N - 2)T_s) \\
+ x[N - 1]p(t - (N - 1)T_s)
\]

\[
x(t) = \sum_{n=0}^{N-1} x[n]p(t - nT_s)
\]
• Rectangular pulse in general cannot give smooth output
4. Sampling and Aliasing

- **Next try triangular pulse**
- **Let** $p(t) = \begin{cases} 1 & \text{for } -\frac{T_s}{2} \leq t \leq \frac{T_s}{2} \\ 0 & \text{else} \end{cases}

\[ x(t) = \sum_{n=0}^{N-1} x[n] p(t - nT_s) \]
• Triangular pulse in general gives better but not the best output
4. Sampling and Aliasing

- The best pulse is sinc pulse

\[ p(t) = \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\pi t / T_s} \]

\[ x(t) = x[0]p(t) + x[4]p(t - 4T_s) \]
Sinc pulse gives the best result
• However, the length of a sinc pulse is infinitely long
• Cannot be implemented exactly
• Low pass filter using sinc pulse is the so-called **ideal low pass filter**, it has rectangular bandwidth
Exercise

A signal can be represented by the following formulation

$$x(t) = \left[10 + 4 \cos(2\pi(2000)t)\right]\cos(2\pi(10000)t)$$

• Sketch the two-sided spectrum of this signal
• Is that signal periodic? If so, what is the period?
• What is the Nyquist sampling frequency of this signal?