5. Fourier Transform and Spectrum Analysis
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Spectrum of Non-periodic Signals

- Fourier series help us to find the spectrum of periodic signals
  ![Fourier Series Example]

- Most signals are not periodic
  - Speech, audio, etc.
  ![Non-periodic Signal Example]

- Need another tool to find the spectrum of non-periodic (aperiodic) signals
  ⇒ Fourier Transform
Fourier Transform of Discrete-time Signals

- Let $x(t)$ be an aperiodic continuous-time signal, $x[n]$ is the samples of $x(t)$ such that:
  \[ x[n] = x(nT_s) \]
- The spectrum of $x[n]$ is given by:
  \[ X_p(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT_s} \text{ or } X_p(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \]
Aperiodic Signals have Periodic Spectrum

• It is interesting to note that $X_p(\omega)$ is periodic since

$$X_p(\hat{\omega} + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\hat{\omega} + 2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega}n} e^{-j2\pi nk} = X_p(\hat{\omega})$$
Signal Processing Fundamentals – Part I
Spectrum Analysis and Filtering

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\[ X_p(\hat{\omega}) \]

\[ \hat{\omega} = \omega T_s \]
• If \( x(t) \) has a spectrum of \( X(\omega) \) and \( x[n] = x(nT_s) \) has a spectrum of \( X_p(\omega) \), it can be shown that

\[
X_p(\hat{\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\hat{\omega} + 2\pi k) \quad -\infty \leq \omega \leq \infty
\]

\[
= \ldots + \frac{1}{T_s} X(\hat{\omega} - 2\pi)
\]

\[
+ \frac{1}{T_s} X(\hat{\omega})
\]

\[
+ \frac{1}{T_s} X(\hat{\omega} + 2\pi) + \ldots
\]
Ideal low pass filter

Time Domain

Frequency Domain

Ideal low pass filter

\[ \hat{\omega} = \omega T_s \]
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Time Domain

- $x(t)$
- $x[n]$

Frequency Domain

- $X(\omega)$
- $X_p(\hat{\omega})$

- $\hat{\omega} = \omega T_s$
- $\hat{\omega} = \omega T_s$

$\omega$ vs. $t$

$\omega$ vs. $\omega$
5. Fourier Transform and Spectrum Analysis

**Time Domain**

\[ x(t) \]

\[ x[n] \]

\[ x'(t) \]

**Frequency Domain**

\[ X(\omega) \]

\[ X_p(\hat{\omega}) \]

\[ X'(\omega) \]

\[ \hat{\omega} = \omega T_s \]
• If the signal has frequency components beyond $|\pi|$, after sampling, these frequency components will affect the other replicas in the spectrum.

• Even with an ideal low pass filter, the original signal cannot be reconstructed. This is the so-called alias effect.

• Restate the Shannon Sampling Theorem for general aperiodic signals:

$$\hat{\omega} \leq |\pi| \Rightarrow 2\pi f_{\text{max}} T_s \leq |\pi|$$

$$\Rightarrow 2\pi f_{\text{max}} \leq |\pi f_s| \text{ or } f_s \geq 2f_{\text{max}}$$
A continuous-time aperiodic signal \( x(t) \) with frequencies no higher than \( f_{\text{max}} \) can be reconstructed exactly from its samples \( x[n] = x(nT_s) \) if the samples are taken at a rate \( f_s = 1/T_s \) that is greater than \( 2f_{\text{max}} \).
Real Examples

Time Domain

Frequency Domain

Resulted signal

Ideal low pass filter
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Time Domain

Frequency Domain

Resulted signal

Ideal low pass filter
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Time Domain

Frequency Domain

Resulted signal

Ideal low pass filter
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How to Solve Aliasing Problems?

1. Increase the sampling rate such that \( f_s \geq 2f_{\text{max}} \)

2. Use anti-aliasing filter first

Pre-filter the input signal such that it will never has frequency components beyond \(|\pi|\)
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**Time Domain**

- $x(t)$
- $x[n]$
- $0 \rightarrow T_s^n$

**Frequency Domain**

- $X(\omega)$
- $-B \rightarrow 0 \rightarrow B$
- $X_p(\hat{\omega})$
- $-2\pi \rightarrow 0 \rightarrow 2\pi$

$\hat{\omega} = \omega T_s$

**Anti-aliasing filter**
With anti-aliasing filter

Ideal low pass filter

Without anti-aliasing filter

Ideal low pass filter

Look better
Hear the effect!

Original 44.1kHz sampling

8kHz sampling with aliasing

8kHz sampling with anti-aliasing filter
Discrete Fourier Transform

- Spectrum of aperiodic discrete-time signals is periodic and continuous
- Difficult to be handled by computer
- Since the spectrum is periodic, there’s no point to keep all periods – one period is enough
- Computer cannot handle continuous data, we can only keep some samples of the spectrum
- Interesting enough, such requirements lead to a very simple way to find the spectrum of signals

⇒ Discrete Fourier Transform
• Recall the Fourier transform of an aperiodic discrete sequence

\[ X_p(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \]

• Assume \( x[n] \) is an aperiodic sequence with \( N \) values, i.e. \( \{x[n] : n = 0, 1, ..., N-1\} \)

\[ X_p(\hat{\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\hat{\omega}n} \]
If we are now interested only in $N$ equally spaced frequencies of 1 period of the Fourier spectrum, i.e.

$$X[k] = X_p\left(\frac{k \cdot 2\pi}{N}\right) \quad k = 0, 1, \ldots, N - 1$$

If $N = 13$
Now if we want to compute the value of these N frequencies,

\[ X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{k \cdot 2\pi}{T_s N}\right)nT_s} \]

\[ W_{N}^{nk} = e^{-j2\pi nk / N} \]

\[ = \sum_{n=0}^{N-1} x[n]e^{-jk \cdot 2\pi n / N} = \sum_{n=0}^{N-1} x[n]W_{N}^{nk} \]

for \( k = 0, 1, \ldots, N-1 \)

Discrete Fourier Transform
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- Discrete Fourier Transform (DFT) is exactly the output of the Fourier Transform of an aperiodic sequence at some particular frequencies.

- Inherently periodic since $X[k+N] = X[k]$, although we always only consider one period of $X[k]$.

\[
X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}
\]

\[
X[k+N] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi n(k+N)/N}
\]

\[
= \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} e^{-j2\pi n} = X[k]
\]
• If we know $X[k]$, we can reconstruct back the signal $x[n]$ via the inverse discrete Fourier transform

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk / N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk} \quad \text{for } n = 0, 1, \ldots, N-1$$

Inverse Discrete Fourier Transform
It can be proven as follows:

\[
\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk2\pi n/N} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x[m] e^{-jk2\pi m/N} e^{jk2\pi n/N}
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x[m] e^{jk2\pi(n-m)/N}
\]

\[
= \frac{1}{N} \sum_{m=0}^{N-1} x[m] \left[ \sum_{k=0}^{N-1} e^{jk2\pi(n-m)/N} \right]
\]

\[
= x[n] \quad \text{for} \quad n = 0, 1, \ldots, N - 1
\]
Although DFT gives exact frequency response of a signal, sometimes it may not give the desired spectrum.

Example

One period of $X_p(\hat{\omega})$

$X[k]$ if $N = 10$

So different from $X_p(\hat{\omega})$
• Need improved resolution

• Achieve by **padding zero** to the end of \( x[n] \) to make \( N \) bigger

\[
x[n] = \{1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ \ldots \ 0\}
\]

40 zeros
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![Graph of |X(\omega)|](image)

![Graph of |X(\frac{2\pi k}{N})|](image)

\(N = 50\)
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\[ \left| X(\omega) \right| \]

\[ \left| X\left(\frac{2\pi k}{N}\right) \right| \]

\( N = 100 \)
Exercise

Given that $x[n]$ is defined in the following figure, determine its spectrum using DFT with $N = 4$.