Angular Modulation

Reference
– Chapter 5.1, Carlson, Communication Systems

Introduction
– AM
  • modulated spectrum
    – translated message spectrum
  • transmission bandwidth
    – \( \leq 2 \times \) message bandwidth
  • S/N (at the receiver)
    – can be improved only by increasing the transmitted power
Angular Modulation

- Angular Modulation
  - Frequency Modulation (FM)
  - Phase Modulation (PM)
  - transmission bandwidth $\geq 2 \times$ message bandwidth
- Nonlinear

$$f(t) = \cos 200\pi t + \cos 300\pi t$$

$$g(t) = f^2(t)$$

$$g(t) = \left[ \cos 200\pi t + \cos 300\pi t \right]^2$$

$$= \cos^2 200\pi t + \cos^2 300\pi t + 2 \cos 200\pi t \cos 300\pi t$$

$$= \frac{1}{2} \cos 400\pi t + \frac{1}{2} + \frac{1}{2} \cos 600\pi t + \frac{1}{2} + \cos 500\pi t + \cos 100\pi t$$

- The signal-to-noise ratio can be improved by trading with the bandwidth.
Angular Modulation

Theory
Consider \( c(t) = A \cos \theta(t) \) where \( \theta(t) = [\omega_{c}t + \phi(t)] \)
- a sinusoidal signal with constant envelope but \textbf{time-varying phase}
- If \( \theta(t) \) contains the modulating signal \( f(t) \), this type of modulation is called \textbf{angle modulation}.

PM
\[
\theta(t) \propto f(t)
\]
- \( \theta(t) \) : instantaneous phase
- The modulated signal is \( f_{PM}(t) = A \cos[\omega_{c}t + k_{p}f(t) + \theta_{o}] \)
  - \( \omega_{c} \): carrier frequency
  - \( k_{p}, \theta_{o} \): constants
  - The \textbf{instantaneous frequency} of this phase-modulated signal is
\[
\omega_{i} = \frac{d\theta}{dt} = \omega_{c} + k_{p} \frac{d}{dt} f(t)
\]
Angular Modulation

**FM**

\[ \omega_i(t) \propto f(t) \]

\[ f_{FM}(t) = A \cos \left[ \omega_c t + k_f \int_0^t f(\tau) d\tau + \theta_o \right] \]

\( \omega_c \): carrier frequency \( k_f, \theta_o \): constants

\[ \omega_i = ? \]

**Example**

\[ f(t) \]

\[ f_{PM}(t) ? \]

carrier signal

\[ f_{FM}(t) \]

FM.4
A FM signal can be expressed as

\[ f_{FM}(t) = A \cos(\omega_c t + k_f \int_0^t f(\tau) d\tau + \theta_o) \]

\[ = \text{Re} \left\{ A e^{j(\omega_c t + k_f \int_0^t f(\tau) d\tau + \theta_o)} \right\} \]

\[ = \text{Re} \left\{ A e^{j(\omega_c t + \theta_o)} e^{jk_f \int_0^t f(\tau) d\tau} \right\} \]

\[ = \text{Re} \left\{ A e^{j(\omega_c t + \theta_o)} e^{jk_f g(t)} \right\} \quad \text{where } g(t) = \int_0^t f(\tau) d\tau \]

\[ = \text{Re} \left\{ A e^{j(\omega_c t + \theta_o)} \left[ 1 + jk_f g(t) - \frac{1}{2!} k_f^2 g^2(t) - \ldots \right] \right\} \]

\[ \therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \]

- From this expression, we conclude that FM is nonlinear unless

\[ |k_f f(t)| \ll 1 \]
Narrowband FM

Consider a sinusoidal modulating signal \( f(t) = a \cos \omega_m t \)

\[
f_{FM}(t) = A \cos \left[ \omega_c t + k_f \int_0^t f(\tau) d\tau + \theta_o \right]
\]

\[
= A \cos \left[ \omega_c t + k_f \int_0^t a \cos \omega_m \tau d\tau \right] \text{ let } \theta_o = 0
\]

\[
= A \cos \left[ \omega_c t + \Delta \omega \int_0^t \cos \omega_m \tau d\tau \right] \Delta \omega = ak_f \text{: peak frequency deviation}
\]

\[
= A \cos \left[ \omega_c t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t \right]
\]

\[
= A \cos \left[ \omega_c t + \beta \sin \omega_m t \right] \quad \beta = \frac{\Delta \omega}{\omega_m} \text{: modulation index}
\]

The FM signal can be rewritten as

\[
A \cos \omega_c t \cos[\beta \sin \omega_m t] - A \sin \omega_c t \sin[\beta \sin \omega_m t]
\]

For small modulation index, we can written as

\[
\therefore f_{NBFM}(t) = A \cos \omega_c t - A \beta \sin \omega_c t \sin \omega_m t
\]

\[
\therefore \cos[\beta \sin \omega_m t] \approx 1 \quad \sin[\beta \sin \omega_m t] \approx \beta \sin \omega_m t
\]

Normally, a criterion for NBFM is \( \beta < 0.2 \)
Example

**AM:** \( f_{AM}(t) = A \cos \omega_c t + m A \cos \omega_c t \cos \omega_m t \)

**NBFM:** \( f_{NBFM}(t) = A \cos \omega_c t - A \beta \sin \omega_c t \sin \omega_m t \)

Let \( A = 1 \), \( m = \beta = 0.1 \), we have
Narrowband FM

– AM
  • the modulation is added in phase with the carrier

\[ m A \cos \omega_c t \cos \omega_m t \quad A \cos \omega_c t \]

– NBFM
  • the modulation is added in quadrature with the carrier

\[ A\beta \sin \omega_c t \sin \omega_m t \quad A \cos \omega_c t \]
Narrowband FM

- Spectral density

\[ F_{\text{NBFM}} (\omega) = F\{A \cos \omega_c t - \beta A \sin \omega_m t \sin \omega_c t\} \]

\[ = F\left\{ A \cos \omega_c t + \frac{\beta A}{2} \cos(\omega_c + \omega_m) t - \frac{\beta A}{2} \cos(\omega_c - \omega_m) t \right\} \]

\[ = \pi A \left[ \delta(\omega + \omega_c) + \delta(\omega - \omega_c) \right] + \frac{\beta \pi A}{2} \left[ \delta(\omega + \omega_c + \omega_m) t + \delta(\omega - \omega_c - \omega_m) t \right] - \frac{\beta \pi A}{2} \left[ \delta(\omega + \omega_c - \omega_m) t + \delta(\omega - \omega_c + \omega_m) t \right] \]

\[ F_{\text{AM}} (\omega) = F\{A \cos \omega_c t + mA \cos \omega_m t \cos \omega_c t\} \]

\[ = \pi A \left[ \delta(\omega + \omega_c) + \delta(\omega - \omega_c) \right] + \frac{m \pi A}{2} \left[ \delta(\omega + \omega_c + \omega_m) t + \delta(\omega - \omega_c - \omega_m) t \right] + \frac{m \pi A}{2} \left[ \delta(\omega + \omega_c - \omega_m) t + \delta(\omega - \omega_c + \omega_m) t \right] \]
Wideband FM

\[ f(t) = a \cos \omega_m t \]

\[ f_{FM}(t) = A \cos[\omega_c t + \beta \sin \omega_m t] \]

\[ = \text{Re}\left\{e^{j\omega_c t} e^{j\beta \sin \omega_m t}\right\} \]

\[ = \text{Re}\left\{e^{j\omega_c t} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_m t}\right\} \]

where \( e^{j\beta \sin \omega_m t} \) is a periodic function with a fundamental frequency of \( \omega_m \)

\[ F_n = \frac{1}{T} \int_{-T/2}^{T/2} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin \xi - n\xi)} d\xi \quad (\text{let } \xi = \omega_m t = \frac{2\pi}{T} t) \]

\[ = J_n(\beta) \]

\( J_n(\beta) \): Bessel function of the first kind of order \( n \) and argument \( \beta \)
Wideband FM

\[ f_{FM}(t) = \text{Re}\left\{ A e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j n \omega_m t} \right\} \]

\[ = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n \omega_m) t \]

\[ F_{FM}(\omega) = \pi A \sum_{n=-\infty}^{\infty} J_n(\beta) [\delta(\omega + \omega_c + n \omega_m) + \delta(\omega - \omega_c - n \omega_m)] \]

- FM signal with sinusoidal modulation has an infinite number of sidebands.

- The spectral density of the sideband at frequency \( \omega_c + n \omega_m \) is proportional \( J_n(\beta) \)
Wideband FM

\[ F_{FM}(\omega) = \pi A \sum_{n=-\infty}^{\infty} J_n(\beta) \left[ \delta(\omega + \omega_c + n\omega_m) + \delta(\omega - \omega_c - n\omega_m) \right] \]
Wideband FM

\( J_n(\beta) \)

- \( J_n(\beta) \) are real valued.
- \( J_n(\beta) = J_{-n}(\beta) \) for \( n \) even
- \( J_n(\beta) = -J_{-n}(\beta) \) for \( n \) odd
Wideband FM

\(AJ_0(\beta)/2\)

- The amplitude of the carrier
- varies with the modulation index
  - depends on the modulating signal
  - contains part of the message information

- The number of significant sideband lines depends on \(\beta\). With \(\beta << 1\) only \(J_o\) and \(J_1\) are significant.
**Bandwidth**

- In most applications, a sideband is significant if its magnitude is equal to or exceed 1% of the unmodulated carrier, i.e.,

\[ |J_n(\beta)| \geq 0.01 \]

- The bandwidth can also be calculated using the following equations
  
  - For large $\beta$,  \( W \approx 2\Delta \omega \)
  
  - For small $\beta$,  \( W \approx 2\omega_m \)
  
  - Carson rule,  \( W \approx 2(\omega_m + \Delta \omega) \)