EIE 441 Advanced Digital communications

MATCHED FILTER

1. Consider the signal $s_i(t)$ shown in Fig. 1.
   
   (a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.
   
   (b) Plot the matched filter output as a function of time.
   
   (c) What is the peak value of the output?

   ![Fig. 1](image)

ERROR RATE OF PAM SYSTEMS

2. In a binary PCM system, symbols 0 and 1 have a priori probabilities $p_0$ and $p_1$, respectively. The conditional probability density function of the random variable $Y$ (with sample value $y$) obtained by sampling the matched filter output in the receiver of Figure 2 at the end of a signaling interval, given that symbol 0 was transmitted, is denoted by $f_Y(y|0)$. Similarly, $f_Y(y|1)$ denotes the conditional probability density function of $Y$, given that symbol 1 was transmitted. Let $\lambda$ denote the threshold used in the receiver, so that if the sample value $y$ exceeds $\lambda$, the receiver decides in favor of symbol 1; otherwise, it decides in favor of symbol 0. Show that the optimum threshold $\lambda_{opt}$, for which the average probability of error is a minimum, is given by the solution of

   \[
   \frac{f_Y(\lambda_{opt} | 1)}{f_Y(\lambda_{opt} | 0)} = \frac{p_0}{p_1},
   \]

![Figure 2](image)

Figure 2. Receiver for baseband transmission of binary-encoded PCM wave using polar NRZ signaling.
3. A binary PCM system using polar NRZ signaling operates just above the error threshold with an average probability of error equal to $10^{-6}$. Suppose that the signaling rate is doubled. Find the new value of the average probability of error.

4. A continuous-time signal is sampled and then transmitted as a PCM signal. The random variable at the input of the decision device in the receiver has a variance of 0.01 volts$^2$. Assuming the use of polar NRZ signaling, determine the pulse amplitude that must be transmitted for the average error rate not to exceed $1 \text{ bit in } 10^8 \text{ bits}$.

5. In this problem, we revisit the PCM receiver of Figure 2, but this time we consider the use of bipolar nonreturn-to-zero signaling, in which case the transmitted signal $s(t)$ is defined by
   - Binary symbol 1: $s(t) = \pm A$ for $0 < t \leq T$
   - Binary symbol 0: $s(t) = 0$, $0 < t \leq T$

   Determine the average probability of symbol error $P_e$ for this receiver assuming that the binary symbols 0 and 1 are equiprobable.

**Intersymbol Interference (Nyquist Channel and Raised Cosine Channel)**

6. An analog signal is sampled, quantized, and encoded into a binary PCM wave. The specifications of the PCM system include the following:

   - Sampling rate = 8kHz
   - Number of representation levels = 64

   The PCM wave is transmitted over a baseband channel using discrete pulse-amplitude modulation. Determine the minimum bandwidth required for transmitting the PCM wave if each pulse is allowed to take on the following number of amplitude levels: 2, 4 or 8.

7. A computer puts out binary data at a rate of 56 kb/s. The computer output is transmitted using a baseband binary PAM system that is designed to have a raised-cosine spectrum. Determine the transmission bandwidth required for each of the following rolloff factors: $\alpha = 0.25, 0.5, 0.75, 1.0$. 

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8. An analog signal is sampled, quantized, and encoded into a binary PCM wave. The number of representation levels used is 128. A synchronizing pulse is added at the end of each code word representing a sample of the analog signal. The resulting PCM wave is transmitted over a channel of bandwidth 12kHz using a quaternary (4-levels) PAM system with raised-cosine spectrum. The rolloff factor is unity.

   a. Find the rate (bit/s) at which information is transmitted through the channel.

   b. Find the rate at which the analog signal is sampled. What is the maximum possible value for the highest frequency component of the analog signal?

9. A binary PAM wave is to be transmitted over a baseband channel with an absolute maximum bandwidth of 75 kHz. The bit duration is 10µs. Find the raised-cosine spectrum that satisfies these requirements.

DUOBINARY CODING AND EQUALIZATION

10. The binary data stream 001101001 is applied to the input of a duobinary system. Construct the duobinary coder output and corresponding receiver output.

11. The unequalized pulse in a PAM system has the following values at sampling times:

\[ p_r(kT_p) = \begin{cases} 
0.2 & k = 1 \\
0.8 & k = 0 \\
0.2 & k = -1 \\
0 & |k| > 1 
\end{cases} \]

(a) Design a three-tap zero forcing equalizer so that the equalizer output is 1 at \( k=0 \) and 0 at \( k=\pm1 \) and \( -1 \).

(b) Calculate the values of the equalized pulse \( p_{eq}(kT_p) \) for \( k = \pm2,\pm3 \).
**Solution**

**1a**

The impulse response of the matched filter is

\[ h(t) = s(T - t) \]

![Diagram of h(t)](image)

**1b**

The matched filter output \( s_o(t) \) is obtained by convolving \( h(t) \) with \( s(t) \), i.e.,

\[ s_o(t) = \int_{-\infty}^{\infty} s(\tau) h(t - \tau) d\tau \]

![Diagram of s_o(t)](image)

**1c**

The peak value of the filter output is equal to \( A^2 T/4 \), occurring \( t = T \).

**2**

The average probability of error is

\[ P_e = p_1 \int_{-\infty}^{\lambda} f_Y(y \mid 1) dx + p_0 \int_{\lambda}^{\infty} f_Y(y \mid 0) dx \]  \hspace{1cm} (1)

An optimum choice of \( \lambda \) corresponds to minimum \( P_e \). Differentiating Eq. (1) with respect to \( \lambda \), we get:

\[ \frac{\partial P_e}{\partial \lambda} = p_1 f_Y(\lambda \mid 1) - p_0 f_Y(\lambda \mid 0) \]
Setting \( \frac{\partial P_e}{\partial \lambda} = 0 \), we get the following condition for the optimum value of \( \lambda \):

\[
\frac{f_Y(\lambda_{opt} \mid 1)}{f_Y(\lambda_{opt} \mid 0)} = \frac{p_0}{p_1}
\]

which is the desired result.

3. \( P_e = 10^{-6} \)

From the graph, \( \sqrt{E_b / N_o} = 3.3 \)

If the signaling rate is doubled, the bit duration is reduced by half. Correspondingly, \( E_b \) is reduced by half.

New \( P_e = 1/2 \operatorname{erfc}(3.3/\sqrt{2}) = 10^{-3} \)

4. The approach is similar to question 3.
   (The variance is \( N_o / 2T_b \))

5. Probability of error for bipolar NRZ signal

   Binary symbol 1: \( s(t) = A \)
   Binary symbol 0: \( s(t) = 0 \)

   Energy of symbol 1 = \( E_b = A^2 T_b \)

   The absolute value of the threshold is \( \lambda = \frac{1}{2} \sqrt{E_b} = \frac{1}{2} \sqrt{A^2 T_b} \).

   Referring to Fig. 1 on the next page, we may write

\[
P(\text{error}|s=\pm A) = \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{\infty} \exp \left[ -\left( y + \frac{\sqrt{E_b}}{\sqrt{N_0}} \right)^2 \right] dy
\]

Let \( z = \frac{y + \sqrt{E_b}}{\sqrt{N_0}} \)
Then,

\[ P(\text{error}|s = -A) = \frac{\lambda + \sqrt{E_b}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\lambda + \sqrt{E_b}} \exp(-z^2) \, dz \]

\[ = \frac{1}{2} \left[ \text{erfc}\left(\frac{1}{2} \frac{E_b}{\sqrt{N_0}}\right) - \text{erfc}\left(\frac{3}{4} \frac{E_b}{\sqrt{N_0}}\right) \right] \]

Similarly, \( P(\text{error}|s = +A) = P(\text{error}|s = -A) \)

\[ P(\text{error}|s = 0) = \frac{2}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N_0}\right) \, dy \]

\[ = \text{erfc}\left(\frac{1}{2} \frac{E_b}{\sqrt{N_0}}\right) \]

The average probability of error is therefore

\[ P_e = P(s = \pm A)P(\text{error}|s = \pm A) + P(s = 0)P(\text{error}|s = 0) \]

The conditional probability density functions of symbols 1 and 0 are given:

\[ P_e = \frac{1}{2} \times \frac{1}{2} \left[ \text{erfc}\left(\frac{1}{2} \frac{E_b}{\sqrt{N_0}}\right) - \text{erfc}\left(\frac{3}{4} \frac{E_b}{\sqrt{N_0}}\right) \right] + \frac{1}{2} \text{erfc}\left(\frac{1}{2} \frac{E_b}{\sqrt{N_0}}\right) \]

\[ = \frac{3}{4} \text{erfc}\left(\frac{1}{2} \frac{E_b}{\sqrt{N_0}}\right) - \frac{1}{4} \text{erfc}\left(\frac{3}{4} \frac{E_b}{\sqrt{N_0}}\right) \]
6. Minimum bandwidth = $1/(2T)$ Hz
   Where $T$ is the pulse duration

   Date rate = $8k \times \log_2 64 = 48$ kbps
   $T_b = 1/$Data rate
   $T = T_b \log_2 M \quad \ldots \text{eq}(1)$
   Where $M$ is the number of amplitude levels

   Therefore, minimum bandwidth = $1/(2T) = 1/(2T_b) = $ Data rate$/2 = 24$ kHz for PAM with two levels.
   Similarly, minimum bandwidth = 12 kHz for PAM with four levels
   Minimum bandwidth = 8 kHz for PAM with eight levels

7. Bandwidth = $W(1+\alpha)$ where $W = 1/(2T_b) = 28$ kHz
   Therefore,
   \[
   \begin{array}{lcc}
   \alpha & \text{bandwidth} \\
   .25 & 35 \text{ kHz} \\
   .5 & 42 \text{ kHz} \\
   .75 & 49 \text{ kHz} \\
   1.0 & 56 \text{ kHz} \\
   \end{array}
   \]

8. (a) For a unity rolloff,
   bandwidth = $1/T$ where $T$ is the pulse duration
   using eq(1), $T_b = T/2$
   Therefore, data rate = $1/T_b = 2/T = 2 \times$ bandwidth = 24 kbps

   (b) no. of bits transmitted for each sample
   = $\log_2$(quantizing levels) + additional bit for synchronization = 7 + 1 = 8 bits
   sampling rate = data rate / no. of bits per sample = 3 kHz

9. Transmission bandwidth = $W(1+\alpha)$. Therefore, $\alpha = 75 \text{ kHz} / [1/(2 \times 10 \mu s)] - 1 = 0.5$
10.  
Binary Sequence: 0 0 1 1 0 1 0 0 1  
Polar representation: -1 -1 1 1 -1 -1 -1 1  
Duobinary coder output -2 0 2 0 0 0 -2 0  
Receiver output -1 1 1 -1 1 -1 -1 1  
Output binary sequence 0 1 1 0 1 0 0 1  

11.  
The equalizer output is  \( y_k = a_{-1}x_{k+1} + a_0x_k + a_1x_{k-1} \). Enforcing the output for \( k = 0, \pm 1 \), we have  
\[
0.8a_{-1} + 0.2a_0 = 0 \\
0.2a_{-1} + 0.8a_0 + 0.2a_1 = 1 \\
0.2a_0 + 0.8a_1 = 0 
\]
Therefore, \( a_{-1} = -0.3571, a_0 = 1.4286, a_1 = -0.3571 \) and  
\[
y_k = -0.3571x_{k+1} + 1.4286x_k - 0.3571x_{k-1} 
\]
For \( k=+2, \ y_k = -0.0714 \)  
For \( k=-2, \ y_k = -0.0714 \)  
For \( k=+3, \ y_k = 0 \)  
For \( k=-3, \ y_k = 0 \)