**Baseband Data Transmission**

Data are sent *without* using a carrier signal.

**Example:** Bipolar NRZ (non-return-to-zero) signaling

- ‘1’ is represented by \[ A \]
- ‘0’ is represented by \[ -A \]

‘1101’ is represented by \[ \]

**Passband Data Transmission**

Data are sent using a carrier signal.

**Example:** PSK (Phase Shift Keying)

- ‘1’ is represented by \[ T_b \]
- ‘0’ is represented by \[ \]
- ‘1101’ is represented by \[ \]
A basic problem in communication is detecting a signal transmitted over a channel that is corrupted by channel noise. The transmitted signal is $g(t)$ and the received signal is $x(t) = g(t) + w(t)$.

**Optimum receiver: matched filter**

**Matched Filter**

A matched filter is a linear filter designed to provide the maximum signal-to-noise power ratio at its output. This is very often used at the receiver.

It is assumed that the receiver has knowledge of the waveform of the signal $g(t)$. 

- **Signal $g(t)$**
- **White noise $w(t)$**
- **Linear time-invariant filter of impulse response $h(t)$**
- **Sample at time $t = T$**
- **Channel**
- **Matched filter**
- **Sampler**
**Matched Filter**

**Signal Power**

Let $G(f)$ and $H(f)$ denoted the Fourier Transform of $g(t)$ and $h(t)$, we have $G_o(f) = H(f)G(f)$

$$\Rightarrow g_o(t) = \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi ft)df \quad \text{(Inverse FT)}$$

The signal power at $T = |g_o(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi ft)df \right|^2$$

**Noise Power**

Since $w(t)$ is white with a power spectral density $N_o/2$, the spectral density function of Noise is

$$S_N(f) = \frac{N_o}{2} |H(f)|^2$$

The noise power $= E[n^2(t)] = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$
**Matched Filter**

**S/N Ratio**

Thus the signal to noise ratio become

\[
\eta = \frac{\left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi fT) df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}
\] .......(1)

Our problem is to find, for a given \( G(f) \), the particular form of the transfer function \( H(f) \) of the filter that makes \( \eta \) at maximum.

---

**Matched Filter**

**Schwarz’s inequality:**

If \( \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx < \infty \) and \( \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx < \infty \)

\[
\left| \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx
\]

This equality holds if and only if, we have \( \phi_1(x) = k\phi_2^*(x) \) where \( k \) is an arbitrary constant, and \( * \) denotes complex conjugation.
Matched Filter

Setting $\phi_1(t) = H(f)$ and $\phi_2(t) = G(f)e^{j2\pi ft}$ and then applying the Schwarz’s inequality to equation (1), we have

$$\eta = \left( \int_{-\infty}^{\infty} |H(f)|^2 \, df - \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df \right)^2$$

$$\Rightarrow \eta \leq \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df \left( |e^{j2\pi ft}| = 1 \right)$$

$$\int \phi_1(x)\phi_2(x)dx \leq \int |\phi_1(x)|^2 \, dx \int |\phi_2(x)|^2 \, dx$$

Therefore,

$$\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 \, df$$

or

$$\eta \leq \frac{2E}{N_0} \quad \text{……(3)}$$

where $E = \int_{-\infty}^{\infty} |G(f)|^2 \, df$ is the input signal energy.
**Matched Filter**

The **maximum S/N** ratio is then obtained when

\[ H(f) = kG^*(f)\exp(-j2\pi fT) \]

\[ \phi_1(x) = k\phi_2^*(x) \]

Taking the inverse Fourier transform of \( H(f) \) we have

\[ h(t) = k\int_{-\infty}^{\infty} G^*(f)\exp[-j2\pi f(T-t)]df \]

and \( G^*(f) = G(-f) \) for real signal \( g(t) \)

\[ h(t) = k\int_{-\infty}^{\infty} G(-f)\exp[-j2\pi f(T-t)]df \]

\[ = kg(T-t) \]

It shows that the impulse response of the filter is the time-reversed and delayed version of the input signal \( g(t) \).

---

**Matched Filter**

**Example:** The signal is a rectangular pulse.

![Diagram of matched filter example](image)

The impulse response of the matched filter has exactly the same waveform as the signal.
**Matched Filter**

The output signal of the matched filter has a triangular waveform.

\[ g_o(t) = g(t) \otimes h(t) \]

In this special case, the matched filter can be implemented using a circuit known as integrate-and-dump circuit.

**Realization of the Matched filter**

The matched filter output is

\[ y(t) = r(t) \otimes h(t) \]

\[ = \int_0^T r(\tau) h(t - \tau) d\tau \]

\[ = \int_0^T r(\tau) g[T - (t - \tau)] d\tau \]

\[ \therefore h(t) = g(T - t) \]

\[ \therefore y(T) = \int_0^T r(\tau) g(\tau) d\tau \]
Error Rate (Binary PAM)

Signaling
- Consider a non-return-to-zero (NRZ) signaling (sometime called bipolar). Symbol 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration.

Noise
- The channel noise is modeled as additive white Gaussian (AWG) noise of zero mean and power spectral density $N_0/2$. In the signaling interval $0 \leq t \leq T_b$, the received signal is

$$x(t) = \begin{cases} + A + w(t) & \text{symbol 1 was sent} \\ - A + w(t) & \text{symbol 0 was sent} \end{cases}$$

- $A$ is the transmitted pulse amplitude
- $T_b$ is the bit duration

Receiver
- It is assumed that the receiver has prior knowledge of the pulse shape, but not its polarity.
- Given the noisy signal $x(t)$, the receiver is required to make a decision in each signaling interval.
Error Rate (Binary PAM)

In actual transmission, a decision device is used to determine the received signal. There are two types of error:

Case 1: Symbol 1 is chosen when a 0 was actually transmitted
Case 2: Symbol 0 is chosen when a 1 was actually transmitted

Example: BSC

\[ x(t) = -A + n(t) \]

The matched filter output \( y(t) \) is

\[ y(t) = \frac{1}{T_b} \int_{0}^{T_b} x(t) dt = -A + \frac{1}{T_b} \int_{0}^{T_b} n(t) dt \]
Error Rate (Binary PAM)

As the noise is white and Gaussian, \( y(t) \) is also Gaussian with the following parameters:

mean of \( y(t) \):

\[
\bar{y} = E(y) = E[-A + \frac{1}{T_b} \int_0^{T_b} n(t)dt] = E[-A] + E[\frac{1}{T_b} \int_0^{T_b} n(t)dt] = -A
\]

Error Rate (Binary PAM)

Variance of \( y(t) \):

\[
\sigma_y^2 = E[(y - \bar{y})^2] = \frac{N_0}{2T_b}
\]

(Proof refers to p.254, *S. Haykin, Communication Systems*)
The probability density function of a Gaussian distributed signal is

\[
f_y(y|0) = \frac{1}{\sqrt{2\pi\sigma_y}} \exp\left(-\frac{(y - \overline{y})^2}{2\sigma_y^2}\right)
\]

Therefore, the conditional probability density function of the random variable \(y\), given that 0 was sent, is

\[
\therefore f_y(y|0) = \frac{1}{\sqrt{\pi N_0 / T_b}} \exp\left(-\frac{(y + A)^2}{N_0 / T_b}\right)
\]

Error Rate (Binary PAM)

Let \(p_{10}\) denote the conditional probability of error, given that symbol 0 was sent

- This probability is defined by the shaded area under the curve of \(f_y(y|0)\) from the threshold \(\lambda\) to infinity, which corresponds to the range of values assumed by \(y\) for a decision in favor of symbol 1

\[p_{10} = P(y > \lambda)\]
Error Rate (Binary PAM)

The probability of error is

\[ P_{10} = P(y > \lambda | \text{Symbol 0 was sent}) \]

\[ = \int_{\lambda}^{\infty} f_y(y|0) dy \]

\[ = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y + A)^2}{N_0 / T_b}\right) dy \]

Assuming that symbols 0 and 1 occur with equal probability, i.e. \( P_0 = P_1 = 1/2 \)

If there is no noise, the output at the matched filter will be \(-A\) for symbol 0 and \(A\) for symbol 1. The threshold \( \lambda \) is set to be 0.

---

Error Rate (Binary PAM)

\[ P_{10} = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y + A)^2}{N_0 / T_b}\right) dy \]

Define a new variable \( z = \frac{y + A}{\sqrt{N_0 / T_b}} \) and then \( dy = \frac{N_0}{\sqrt{T_b}} dz \).

\[ \therefore P_{10} = \frac{1}{\sqrt{\pi}} \int_{E_b/N_o}^{\infty} \exp(-z^2) dz \]

where \( E_b \) is the transmitted signal energy per bit, defined by \( E_b = A^2 T_b \).
At this point we find it convenient to introduce the definite integration called complementary error function.

$$\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} \exp(-z^2)dz$$

Therefore, the conditional probability of error

$$P_{10} = \frac{1}{2} \text{erfc} \left( \frac{E_b}{\sqrt{N_0}} \right)$$

( Note: \( \text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_{0}^{u} \exp(-z^2)dz \) and \( \text{erfc}(u) = 1 - \text{erf}(u) \) )

In some literature, Q function is used instead of erfc function.

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp(-\frac{u^2}{2})du$$

$$Q(x) = \frac{1}{2} \text{erfc} \left( \frac{x}{\sqrt{2}} \right) \text{ and } \text{erfc}(x) = 2Q(x\sqrt{2})$$
Case II

Similarly, the conditional probability density function of $y$ given that symbol 1 was sent, is

$$f_y(y|1) = \frac{1}{\sqrt{\pi N_0 / T_b}} \exp\left(-\frac{(y - A)^2}{N_0 / T_b}\right)$$

$$P_{01} = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y - A)^2}{N_0 / T_b}\right)dy$$

By setting $\lambda = 0$ and putting $z = \frac{y - A}{\sqrt{N_0 / T_b}}$, we find that $P_{01} = P_{10}$.

The average probability of symbol error $P_e$ is obtained as

$$P_e = P_{01}P_{10} + P_{11}P_{01}$$

If the probability of 0 and 1 are equal and equal to $\frac{1}{2}$

$$P_e = \frac{1}{2} \text{erfc}\left(\frac{E_b}{\sqrt{N_0}}\right)$$
Example: A polar trinary waveform in additive Gaussian noise with probability density function as shown below. Calculate the net probability of error if the decision thresholds are set at $\pm \frac{A}{4}$.

\begin{align*}
P_0(y) & \\
P_1(y) & \\
P_2(y) & \\
-\frac{A}{2} & \quad -\frac{A}{4} \quad 0 \quad \frac{A}{4} \quad \frac{A}{2}
\end{align*}
As the mean of symbol 0 is \( \bar{y} = -A/2 \) and the variance is
\[
\sigma^2 = \frac{N_0}{2T_b}
\]

\[
P_{e0} = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{-A/4}^{\infty} \exp\left(-\frac{(y + A/2)^2}{N_0 / T_b}\right)dy
\]

Put \( z = (y + A/2)/\sqrt{N_0 / T_b} \)

\[
P_{e0} = \frac{2}{\sqrt{\pi}} \int_{E_b / 4N_o}^{\infty} \exp(-z^2)dz = \text{erfc}\left(\sqrt{E_b / 4N_o}\right)
\]

Error Rate (Binary PAM)

\[
P_{e1} = \frac{1}{\sqrt{\pi N_0 / T_b}} \left[ \int_{-A/4}^{-A/4} \exp\left(-\frac{y^2}{N_0 / T_b}\right)dy + \int_{A/4}^{\infty} \exp\left(-\frac{y^2}{N_0 / T_b}\right)dy \right]
\]
\[
= 2 \text{erfc}\left(\sqrt{E_b / 4N_o}\right)
\]

\[
P_{e2} = P_{e0} = \text{erfc}\left(\sqrt{E_b / 4N_o}\right)
\]

If \( P_0 = P_2 = P_3 = 1/3 \)

\[
P_e = P_{e0}P_0 + P_{e1}P_1 + P_{e2}P_2 = \frac{4}{3} \text{erfc}\left(\sqrt{E_b / 4N_o}\right)
\]