Image restoration by regularisation in uncorrelated transform domain


Abstract: Conventional spatially adaptive regularised image restoration schemes weight the amount of regularisation according to the spatial content of an image. The authors first separately decorrelate the signals under analysis into uncorrelated components and then weight the amount of regularisation performed to these components accordingly. The proposed approach works better than conventional schemes, especially in edge regions.

1 Introduction

In image restoration, an image degradation process can be generally formulated by \( y = Hx + n \), where \( x \) and \( y \) are the lexigraphically ordered original and degraded images, \( n \) is a noise vector and \( H \) represents a linear degradation operator [1].

Solving this equation directly to obtain the solution \( \hat{x} \) from the observable \( y \) is not feasible as it is basically an ill-posed problem [2]. Restoration methods based on regularisation theory [3] are widely used instead, since they can successfully replace the ill-posed problem by a well-posed problem. The constrained optimal method [4] is the simplest methodology to realise regularisation. In this method, an algebraic objective function of \( \hat{x} \) is defined based on different constraint sets. The solution is then obtained by minimising the objective function with respect to \( \hat{x} \).

In general, two constraints are used. One of them tries to keep the solution faithful to the information provided by the observed version \( y \), which is usually given as \( \|y - H\hat{x}\|^2 < e \), while the other one tries to keep the solution faithful to the a priori information about the original image, which can be generally given as \( \|L(\hat{x} - \hat{x})\|^2 < e \). Here, \( \hat{x} \) represents our existing knowledge about the solution and \( L \) is a linear operator. The bounds \( e \) and \( e \) respectively, describe how 'faithful' the possible solution should be to the observed information and the a priori information. In other words, their values indicate the relative significance of the constraints to the solution and hence should be used to weight the contribution of the constraints in constructing the objective function [5]. The objective function derived from this idea is given as \( J = \|y - H\hat{x}\|^2 + \alpha \|L(\hat{x} - \hat{x})\|^2 \), where \( \alpha = e/e \).

By probing further, one can see that, in fact, different elements of \( y - H\hat{x} \) make different contributions to the error function \( \|y - H\hat{x}\|^2 \) in fulfilling the constraint \( \|y - H\hat{x}\|^2 < e \). To obtain a good solution \( \hat{x} \), their contribution should also be weighted. A similar case happens when the elements of \( L(\hat{x} - \hat{x}) \) are investigated. By taking these factors into account, the objective function should be modified to be

\[
J = \|y - H\hat{x}\|^2 + \alpha \|L(\hat{x} - \hat{x})\|^2.
\]

Here \( \|y\|^2 \) and \( \|\|^2 \) denote weighted norms. This generalised formulation describes almost all spatially adaptive regularised restoration methods reported in the literature [6–10]. In general, these methods differ in their ways of evaluating \( R \) and \( S \). Both \( R \) and \( S \) are usually oversimplified to be diagonal matrices in these methods.

If introducing weighting factors to weight the contribution of different elements is the right direction to improve the restoration performance, the objective function given as eqn. 1 is obviously not the ultimate solution. Consider the objective function (eqn. 1) again. Since \( R \) and \( S \) are diagonal matrices, each element of \( y - H\hat{x} \) and \( L(\hat{x} - \hat{x}) \) is weighted separately. This implies that the elements are considered to be independent of each other. This is obviously not true, since adjacent image pixels in an image are highly correlated. By using such a simplified weighting approach, the weighting effect of different weighting factors may counteract each other and hence not be able to provide a good restoration result effectively.

To solve this problem, it is proposed that \( y - H\hat{x} \) and \( L(\hat{x} - \hat{x}) \) be considered as two different signals and that these be decomposed separately into a number of uncorrelated channels by transforms for weighting. By doing this, two advantages can be gained. First, it is easier to determine the weighting factor for a particular channel, because uncorrelated channels do not interact with each other. They are isolated and easier to handle. Weighting a particular channel will not affect the other channels. The second advantage also comes from the decorrelation property of the image transform. In practical circumstances, one has to estimate the weighting factors from either the distorted image or the a priori information, so there must be some estimation errors. The less correlated are the channels, the less sensitive is the restoration result to the estimation errors in the channels.

In this paper, based on the idea described above, the transform theory [11] is used to decorrelate images into uncorrelated transform components and these components are then weighted according to their variances. This
approach can improve the restoration performance. In particular, it has been found in the experiments that the details of the edge regions of restored images can be improved greatly using the proposed approach while the other conventional spatially adaptive approaches cannot achieve this [6–10].

Note that some papers consider a distorted image as a multichannel signal and restore it in the frequency domain [12]. However, their motivations and implementations are different from those of this paper. Generally speaking, decorrelating the signal before weighting is not its basic concern. In these approaches, the discrete Fourier transform (DFT) is typically used to decompose the distorted image into a number of frequency channels for subsequent restoration. Since DFT components are still correlated, these approaches can be considered as simultaneously performing spatial weighting schemes on a number of subband images.

2 Algorithm

Suppose  is a vector of random variables. The value of each random variable is of a certain uncertainty but its statistical characteristics are known or can be estimated. Without loss of generality, it is assumed that  and  where  is the expectation operator and  denotes the zero vector. Note that, since its statistical characteristics are known,  can always be zero-meaned. Assume that  is the unitary Karhunen–Loeve transform (KLT) of  [13]. Then  can completely decorrelate  and  is a diagonal matrix. The th diagonal element of the matrix, denoted as  where  is the th element of , indicates the variance of  with respect to the other elements of . This information can hence be used to weight the contribution of each element of  to . Specifically, the weighting factor should be proportional to . If  is the a priori information we know about  then  and can be easily determined as . Based on the idea described, the objective function can be given as

\[
J = \|T_1(y - H\hat{x} - M_1)\|^2 + \alpha \|T_2[L(\hat{x} - \hat{x}) - M_i]\|^2, \quad (2)
\]

where  and  are the KLTs for  and  respectively. Here,  and  are the bounds of  and  respectively. The weighting matrices  and  are diagonal matrices intrinsically and their th diagonal elements can be determined as

\[
r_i = \frac{1}{\langle E[T_1(y - H\hat{x} - M_1)(y - H\hat{x} - M_1)^{T}]\rangle_{ii}}, \quad (3)
\]

\[
s_i = \frac{1}{\langle E[T_2[L(\hat{x} - \hat{x}) - M_i][L(\hat{x} - \hat{x}) - M_i]^{T}]\rangle_{ii}}, \quad (4)
\]

Note eqns. 2–4 provide the general formulations for restoring a degraded image.

Now consider the case when a smoothness constraint is applied. In such a case, one can let  where  is a spatial 2-D highpass Laplacian filter represented in matrix form. As  theoretically contains no low-frequency component, it is assumed that  in order to simply the analysis. In addition, if  is a zero-mean white noise of variance  it can be assumed that

\[
E[y - H\hat{x}][y - H\hat{x}]^{T} = \sigma_y^2 I \quad \text{and} \quad M = E[y - H\hat{x}] = 0.
\]

This implies that  and  hence, the objective function (eqn. 2) can be simplified as

\[
J = \frac{1}{\sigma_y^2}\|y - H\hat{x}\|^2 + \alpha \|T_2[C\hat{x}]\|^2 \quad (5)
\]

The minimisation of  with respect to  results in the normal equation

\[
(H^{T}H + \sigma_y^2 C^{T}T_2C)\hat{x} = H^{T}y \quad (6)
\]

In general,  cannot be evaluated directly from this equation as it requires the inversion of a huge matrix. An alternative approach is to use a steepest-descent algorithm to approximate  iteratively [6]. This approach leads to the iterative equation

\[
\hat{x}_{k+1} = \hat{x}_k + \beta[H^{T}(y - H\hat{x}_k) - \sigma_y^2 C^{T}T_2C\hat{x}_k] \quad (7)
\]

\[
\hat{x}_0 = \beta H^{T}y \quad (8)
\]

where  is the estimate of  at the th iteration. The iteration converges if  satisfies the condition  where  is the largest eigenvalue of the matrix

\[
H^{T}H + \sigma_y^2 C^{T}T_2C.
\]

The weighting matrix  can be estimated at each iteration based on the available form of the restored image  . In particular, by substituting the assumptions mentioned earlier into eqn 4, one obtain

\[
s_i = \frac{1}{\langle E[T_2[C\hat{x}](T_2[C\hat{x}])^{T}]\rangle_{ii}} \quad (9)
\]

Note that  is in fact the variance of the th element of  . In practice, it is estimated with the ensemble  where  is an integer parameter which defines the size of the ensemble and  denotes the shift version of  obtained by shifting all its elements  steps up and  steps right in the spatial domain. Specifically, its estimated value  is given as

\[
\theta_i = \frac{1}{(2d + 1)^2} \sum_{m=-d}^{d} \sum_{n=-d}^{d} [T_2[C\hat{x}](T_2[C\hat{x}])^{T}]_{mn} \quad (10)
\]

where

\[
M_c = \frac{1}{(2d + 1)^2} \sum_{m=-d}^{d} \sum_{n=-d}^{d} [C\hat{x}](C\hat{x})^{T} \quad (11)
\]

In contrast to the approaches which make use of local spatial variance [6–8], the proposed approach approximates weighting factors in the transform domain. As was mentioned in Section 1, it would be helpful to obtain a better restoration result because of the lower sensitivity to approximation error in the transform domain.

The value of  could fluctuate violently from  to  so  may not be stable if one directly lets  be  with eqn 9. To make  stable, the equation  is used instead to confine  in the interval 0,1]. The parameter  is a tuning parameter that can be adjusted experimentally to make the weighting effect be able to provide a good restoration result from the human visual point of view.

As a final remark, note that sometimes an approximation of the KLTs involved in the proposed scheme is required for two practical reasons: first, it is computationally very difficult to determine a KLT kernel of large size; secondly, even though the KLT kernels are given, the computational
complexity of performing them to the images is very high [13]. In practice, the discrete cosine transform (DCT) is used instead to decorrelate the unstacked image \( Cx \). This is because, according to the image-transform theory, an image can typically modelled as a highly correlated 2-D Markov-I signal and the DCT is asymptotically equivalent to the KLT in decorrelating such signals of this type [13]. Other reasons for using the DCT are that there are a number of fast algorithms for its realisation and its realisation complexity is much lower than that of the KLT.

3 Simulation studies

An experiment was first carried out to verify that the DCT was a good approximation to the KLT in decorrelating \( Cx \). In this experiment, four standard images, namely, ‘Lena’, ‘Cameraman’, ‘House’ and ‘Germany’, were filtered with operator \( C \) and then partitioned into a number of subimages of size 8 \( \times \) 8 to form a set \( \Gamma_{Cx} \). Then, based on this set of subimages, the correlation coefficients between any two different pixels of \( Cx \) were computed and plotted against their magnitude order. The magnitude of the correlation coefficients was plotted in descending order. Similarly, by using the set \( \Gamma_{C'y} \), where \( T \) is any particular transform operator, the curve reflecting the correlation among the elements of \( TCx \) is obtained. Fig. 1 shows the experimental result. One can easily observe that performing a transform can definitely decorrelate \( Cx \) and the decorrelation performance of the DCT is the best among the transforms.

Simulations were then performed to evaluate the performance of the proposed restoration scheme through a few deburring examples. In particular, it was hoped to find out whether weighting decorrelated components is more effective than weighting correlated components in providing a good image restoration performance. To achieve this, the scheme proposed in [6] was realised as well for comparison. These two schemes are more or less the same except that the proposed scheme weights the components after decorrelating them while that [6] does not. Hereafter, they are referred to as non-spatially adaptive weighting (NAW) and spatially adaptive weighting (SAW) schemes, respectively.

In the realisation of the proposed scheme, three subshemes were simulated for comparison. In the first subsheme, the decorrelation transform was approximated with periodic 8 \( \times \) 8 two-dimensional DCT transform kernels. Specifically, to decorrelate the unstacked image \( Cx \), it was first partitioned into a number of non-overlapped subimages of size 8 \( \times \) 8 and an 8 \( \times \) 8 DCT was then performed on each of them. In the second and the third subshemes, 256 \( \times \) 256 DCT and DFT were, respectively, exploited to decorrelate the unstacked image \( Cx \).

As for the realisation of the SAW scheme [6], the solution was obtained by the following iterative equations:

\[
\hat{x}_{k+1} = \hat{x}_{k} + \beta_{k} [H^{T}(y - H\hat{x}_{k}) - \alpha_{k} C^{T}S'C\hat{x}_{k}] \tag{12}
\]

\[
\hat{x}_{0} = \beta_{0} H^{T}y \tag{13}
\]

Here, \( S \) is a diagonal matrix whose \( i \)th diagonal element is given as \( \theta_{i} = 1/(1 + \kappa\theta_{i}) \), where \( \theta_{i} \) is the local spatial variance of the corresponding pixel. In particular,

\[
\theta_{i} = \frac{1}{(2d + 1)^{2}} \sum_{m=-d}^{d} \sum_{n=-d}^{d} (\hat{x}_{i})^{<m,n>}_{\alpha} - M_{i}
\]

\[
M_{i} = \frac{1}{(2d + 1)^{2}} \sum_{m=-d}^{d} \sum_{n=-d}^{d} (\hat{x}_{i})^{<m,n>}_{\alpha}
\]

For fairness of comparison, the various parameters in the NAW and SAW schemes are selected in such a way that the only difference between the two schemes is that the former weights the image components after decorrelating them while the latter does it without decorrelation. More specifically, for the realisation of SAW scheme, one obtains \( \alpha_{k} = \alpha_{0}^{2}/10\|Cx\|^{2}_{\alpha} \), \( \beta_{0} = 0.67 \), \( \kappa = 0.05 \), and \( d = 1 \). As for the realisation of the NAW scheme, one obtains \( \alpha_{k} = \alpha_{0} \), \( \beta = \beta_{0} \), \( \kappa = 0.05 \), and \( d = 1 \). It can be seen that, after substituting corresponding sets of parameters into eqns. 7, 8 12 and 13, the contributions of the two sets of parameters to their corresponding schemes are effectively identical. The termination rule for the two schemes was \( \|x_{k+1} - \hat{x}_{k}\|_{\alpha}/\|\hat{x}_{k}\| < 10^{-6} \) during the simulation. The schemes converged to their solutions in all of the performed simulations.

Fig. 2a shows the original image of ‘Cameraman’. In the experiment, it was first defocused with circle of confusion equal to five pixels. White noise was then added resulting in a SNR of 20 dB, where SNR was defined as SNR = 10 log(variance of signal/variance of noise). Fig. 2b shows this distorted image. Figs. 2c and 2d-f are, respectively, the restoration results of the SAW scheme and the NAW schemes.

In general, the SAW scheme provides a poorer restoration result around the edge regions compared with all NAW schemes. This is because the local spatial variance in an edge region is typically larger than that in a smooth region. A SAW scheme will accordingly adapt to it and try not to suppress the noise in the region. This results in a noisy edge region.

Among the three NAW subshemes, that using the DFT for decorrelation provided the poorest result as the decorrelation performance of the DFT is poorer than that of the DCT. One can easily observe the pattern noise in Fig. 2d.
Fig. 2  Restoration performance of various schemes on noisy defocus-blurred image

a Original  d NAW-DFT256
b Degraded  e NAW-DCT256
c SAW       f NAW-BDCT
Fig. 3  Restoration performance of various schemes on noisy motion-blurred image

a Original    d NAW-DFT256
b Degraded    e NAW-DCT256
c SAW         f NAW-BIDCT
As for the other two subschemes, that using block-based DCT provided a better restoration result in terms of the SNR improvement. Images are actually not stationary signal. Use of block-based transform enables the weighting matrix to adapt to the local characteristics of an image. Use of a single DCT to decorrelate an image is impossible to achieve this. However, there may be some visible blocking effect in the restored image if a block-based DCT is used.

Similar findings can be obtained in restoring motion-blurred images. Fig. 3b shows several magnified portions of a noisy and motion-blurred version of ‘Lena’. Its original was first blurred by horizontal motion blur over nine pixels. Noise was then added to achieve a SNR of 20dB. Figs. 3c–f are the corresponding restoration results obtained with the four schemes. One can see that the NAW schemes, especially that using block-based DCT for decorrelation, can provide better restoration results than the SAW scheme. Table 1 summarises the objective results of the experiments, for comparison.

To cast light on the differences in the two weighting schemes, two sets of illustrations are provided showing the evolution of the deblurring results obtained with the SAW and NAW schemes. They are shown in Figs. 4 and 5, respectively. The first column of each of these Figures
Fig. 5 Restored images, restored terms, and regularisation terms produced with NAW-BDCT scheme after different numbers of iterations

Top to bottom: Number of iterations $k = 5, 10, 30$ and $60$
Left column: Restored image $\hat{x}_k$
Middle column: Restoration term $H^T(y - HS\hat{x}_k)$
Right column: Regularisation term $C^TST\hat{C}_k$

depicts a number of restored images $\hat{x}_k$ obtained after $k$ iterations. The middle columns show the corresponding restoration terms $(H^T(y - HS\hat{x}_k))$. The function of this restoration term is to reconstruct the high-frequency components of the original image. One can observe that each of the restoration terms contains the amplified noise as an undesirable artifact. Note that, in early stage of the iteration, the noise amplification is not so severe, but the severity is increased with the number of iterations performed. The last column of each of the Figures depicts the corresponding regularisation terms ($C^TSC\hat{C}_k$ for the SAW scheme and $C^TST\hat{C}_k$ for the NAW scheme). The function of this term is to suppress the noise amplified by the restoration term. Observe that, for the SAW scheme, the regularisation term suppresses the noise only in the smooth regions. This causes the amplified noise to stay around the edge regions. For the NAW scheme, the regularisation term suppresses the noise in both smooth and edge regions. This results in a better restoration of edge information.
Table 1: Summary of the simulation results

<table>
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<tr>
<th>Testing image</th>
<th>Cameraman</th>
<th>Lena</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distortion</td>
<td>Defocus blur 5 x 5, 20 dB SNR</td>
<td>Motion blur 9 x 1, 20 dB SNR</td>
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<tr>
<td>Parameters</td>
<td>$\beta = \beta_0 = 0.67, \sigma_0 = \frac{\sigma^2}{10},</td>
<td>\text{C}</td>
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<tr>
<td>Termination rule</td>
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<td>x_{k+1} - x_k</td>
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<table>
<thead>
<tr>
<th>SNR improvement</th>
<th>Number of iteration</th>
<th>SNR improvement</th>
<th>Number of iteration</th>
</tr>
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<tbody>
<tr>
<td>1.67 (dB)</td>
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<td>3.19 (dB)</td>
<td>51</td>
</tr>
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</tr>
<tr>
<td>1.84 (dB)</td>
<td>34</td>
<td>3.15 (dB)</td>
<td>50</td>
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<tr>
<td>1.83 (dB)</td>
<td>31</td>
<td>3.14 (dB)</td>
<td>47</td>
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</table>

4 Conclusions

Conventional spatially adaptive regularised image restoration schemes weight the amount of regularisation performed to different pixels of the solution according to the spatial content of an image. In this paper, it is suggested instead, the signals under analysis should first be separately decorrelated into a number of uncorrelated components by making use of the image transform theory and then these components should be weighted accordingly. Based on this idea, an effective adaptive iterative restoration algorithm is also proposed. The advantages of the proposed approach over the conventional approaches have been discussed and simulation results have been shown. Simulations verified that weighting decorrelated components could provide a better restoration result than weighting highly correlated image pixels, especially around the edge regions.

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6 References