Novel detection of conics using 2-D Hough planes

P.-K. Ser
W.-C. Siu

Indexing terms: Hough transform, Conic detection

Abstract: The authors present a new approach to the use of the Hough transform for the detection of ellipses in a 2-D image. In the proposed algorithm, the conventional 5-D Hough voting space is replaced by four 2-D Hough planes which require only 90 Kbytes of memory for a 384 × 256 image. One of the main differences between the proposed transform and other techniques is the way to extract feature points from the image under question. For the accumulation process in the Hough domain, an inherent property of the suggested algorithm is its capability to effect verification. Experimental results from the authors' work on real and synthetic images show that a significant improvement of the recognition is achieved as compared to other algorithms. Furthermore, the proposed algorithm is applicable to the detection of both circular and elliptical objects concurrently.

1 Introduction

The Hough transform [1–3] is a famous technique for the detection of analytic and nonanalytic patterns. For the detection of second order analytic curves, a multidimensional Hough domain is required to store the corresponding parameters. In case of the detection of elliptical objects, a 5-D Hough domain is required to store the five geometrical parameters of the ellipse. To reduce the demand of memory for the recognition, different approaches for using multiplanes 2-D Hough space or curve segment histogram [4–9] have been proposed to replace the conventional approaches. Yip et al. [5] proposed a single 2-D Hough space for the detection of elliptical shapes. Although it is able to compress the total memory cost, the capability for the recognition of overlapped objects is lost in Yip's algorithm. For the recognition of circular objects, Ye [10] suggested a simple technique for folding the test image vertically and horizontally to detect the candidate centres. However, it seems to be impractical for the application of the algorithm in the situation with multiple objects because of the interference among objects for the folding operations.

In [11] a point spread function (PSF) for ellipse detection was suggested, which is able to locate the corresponding centre of the ellipse in the image. However, the problem of the generation of the PSF for ellipses with unknown size and orientation is difficult. Instead of using the PSF function to locate the centre of the ellipse, a fast chord bisector technique is suggested by Chan and Siu [12]. The chord bisectors of an ellipse or a circle are all intercepted to the centre of the object. However, the algorithm could have problems for the detection of objects with occluded or with incomplete contours.

Recently, a probabilistic Hough transform [13] was introduced to reduce heavy computation for the recognition of straight lines. Leavers [14] generalised the relationship of probabilistic Hough transform to detect circles and ellipses. Leavers' algorithm maps each set of five image points to a single point on the Hough space. As shown in the experimental results of the paper, the adaptive size of the window for segmentation of the image is an important factor for the efficiency and accuracy of the algorithm. A false location of the proposed connected point, $P_{con}$, using as the reference of the transformation gives great effect to the corresponding parameters of the detected objects. Yoo and Sethi [15] applied pairs of edge pixels to determine the parameters for the corresponding ellipse. To reduce the number of edge pairs for the processing, a preprocessing for the elimination of straight lines is required, that aims to reduce the combination of edge pairs to be considered in the recognition. By using the symmetrical properties of conic boundaries, Tsuji and Matsumoto [16] suggested in 1978 the partition of multidimensional Hough space with a 2-D Hough centre plane and a 1-D Histogram for the curve-fitting process. However, the detection of the candidate centres in Tsuji and Matsumoto's model has been proposed by Wu and Wang [17]. With the risk of missing the detection of concentric objects, Wu's model is able to compress the memory requirement and requires less computation effort.

We suggest a new mapping for each set of three edge pixels to a 2-D Hough plane for centre detection and then estimate the corresponding parameters for some other domains. By using the gradient angle of each edge pixel, we are able to reduce the computation significantly. The conventional 5-D Hough space is replaced by four planes, namely, $(X_{centre}, Y_{centre})$, $(Major, Y_{centre})$, $(Minor, Y_{centre})$ and $(Orientation, Y_{centre})$. After the peak searching for the plane for detecting $(X_{centre}, Y_{centre})$, other parameters of the candidate objects can be easily defined with a 1-D search on the respective domains.
2 Parameterisation of the voting scheme

2.1 Background and basic theory of conic equation
In the Cartesian coordinate system, we may express a second order conic equation using five undetermined coefficients. Eqn. 1 shows a general second order equation which can be used to represent a circle or an ellipse. To determine the corresponding parameters of the conic, five points on the conic boundary are necessary to solve the unknown coefficients in eqn. 1. In the proposed algorithm, a simplified expression for elliptical boundary, as shown in eqn. 2, is applied for the detection. This provides an advantage to the reduction of the number of undetermined coefficients, which can alleviate the problem of accumulation error in the Gaussian elimination and also reduce computation. Thus,

\[
\frac{[(x - x_0) \cos \theta - (y - y_0) \sin \theta]^2}{a} + \frac{[(x - x_0) \sin \theta + (y - y_0) \cos \theta]^2}{b} = 1 \quad (1)
\]

where \((x_0, y_0, a, b, \theta)\) are the parameters of the ellipse.

For the centre of the ellipse locating at the origin of the coordinate system, eqn. 1 can be simplified as shown in eqn. 2.

\[Ax^2 + Hxy + By^2 - 1 = 0 \quad (2)\]

where

\[A = \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right), \quad B = \left( \frac{\sin^2 \theta}{b^2} + \frac{\cos^2 \theta}{a^2} \right)\]

and

\[H = 2 \cos \theta \sin \left( \frac{1}{b^2} - \frac{1}{a^2} \right)\]

To achieve a simple expression for the conic equation, we may assume that the centre of the test conic pattern be at the origin of the coordinate system. Hence, our procedure for using the Hough transform for the detection of conics includes (i) parameterisation voting, (ii) peak searching and (iii) verification of the detected peaks. The process on peak searching and the verification is simple and fast. The local maxima in the Hough space are extracted with a thresholding and the parameters are back projected to the original image to verify the corresponding results. A self checking voting scheme is proposed in the following Section. The characteristic of the self checking enables the elimination of false votes on the Hough space. The major concept of our new scheme for the determination of five parameters is shown as follows.

2.2 Parameterisation of the 2-D Hough planes
Consider a conic segment as shown in Fig. 1. Lines \(L_1\) and \(L_2\) are the tangents at \(P_1\) and \(P_2\), respectively. Let us define the intersection point between \(L_1\) and \(L_2\) to be \(P_z\), then, it is a well known property that the line joining the midpoint of the edge pair \((P_1, P_2)\) and \(P_z\) passes through the centre of the corresponding ellipse.

Referring back to Fig. 1: For any edge pair such as \((P_1, P_3)\) and \((P_2, P_4)\), there is a possible ellipse with its centre which can be defined by the pairs of lines that pass through the intersection of the corresponding tangents and the midpoints. Similarly, by grouping the edge pairs such as \((P_1, P_2)\) and \((P_1, P_3)\), or \((P_1, P_2)\) and \((P_4, P_3)\) one is able to located the same intersection point in the test image, the centres of possible conics (circles or ellipses) can be deduced. However, a main obstacle of sampling randomly edge pixels in triplet form is the processing time for the detection. Let us take an image with \(M\) edge pixels as an example, the computation is proportional to the number of combinations of the edge pixels, \(C_M\), which is too large for a practical realisation. To reduce the number of operations, we restrict the selection of edge pixels according to the following criteria:

(a) \(P_1\), \(P_2\) and \(P_3\) are extracted such that the corresponding conic relationships among them can be calculated easily.

(b) The separation between two selected edge pixels should be as large as possible to reduce the digitalisation error for further process.

Let \((X_{centre}, Y_{centre})\) be the centre detected for the triplet, \(C_1\), \(C_2\) and \(C_3\) in Fig. 2. We use gradient direc-
choose \( C_2 \) and \( C_3 \), while eliminating all possible gradients. Second, the symmetrical property of the conic objects can also be utilised for the improvement of the accuracy during the detection. Since \( C_2 \) and \( C_3 \) are selected to have the same gradient on the boundary, the midpoint \( (X_{mid}, Y_{mid}) \) along \( C_2, C_3 \) is exactly equal to the centre of the object. By a comparison of the consistence whether \( C_1, C_2 \) and \( C_3 \) lie on the boundary of a conic identity. Second, the symmetrical property of the conic choice \( C_1 \) and \( C_2 \), while eliminating all possible gradient elimination method. However, the edge pairs \( C_2 \) and \( C_3 \) are not independent for solving eqn. 2, we have to select another appropriate pixel, say \( C_4 \) or \( C_5 \), shown in Fig. 2 for the calculation of the parameters. For the simplification of the analysis, our explanation is based on pixel \( C_4 \). \( C_4 \) can be defined as the intersection of the line \( L_{12} \) and the boundary of the corresponding ellipse. To define the location of \( C_4 \), let us recall another useful property for an ellipse/circle. The tangent at \( C_4 \) must be equal to the slope of \( C_1, C_2 \). By searching the predefined pixel array according to the slope of \( C_1, C_2 \), we may efficiently extract the position of \( C_4 \) which has the tangent equal to the slope of \( C_1, C_2 \) and lies on the line \( L_{12} \). After getting the edge pixel, \( C_4 \), the unknown coefficients \( A, B \) and \( H \) can be calculated with the triplet \( C_1, C_2 \) and \( C_4 \) to determine the rest of the parameters for the ellipse. Three different 2-D Hough planes \( (a, Y_{centre}), (b, Y_{centre}) \) and \( (\theta, Y_{centre}) \) are used to represent the votes for each parameter. In case \( C_4 \) cannot be located in the test image, an alternative is to substitute \( C_5 \) for \( C_4 \) for the calculation.

For the voting of each triplet in the Hough space, the derivation is based upon some useful conic properties which are applicable for both elliptical and circular objects. Furthermore, we may consider the detection of a circular object as a particular case of ellipse detection. During the recognition, the angle of orientation is always zero for circular object and the calculated length for both of axes, \( a \) and \( b \), are equal in magnitude. Hence, the proposed algorithm is applicable for the detection of both circles and ellipses concurrently.

### 3 Realisation of the voting

As shown above, our proposed recognition technique makes use of three edge pixels in image for each accumulation on the \((X_{centre}, Y_{centre})\) 2-D Hough domain. The process involves also a simple checking of the validity for the corresponding edge pixels, and then the accumulations of parameters in the respective \((a, Y_{centre}), (b, Y_{centre})\) and \((\theta, Y_{centre})\) Hough planes are performed by solving the conic expression. The process is iterated for each suitable group of boundary pixels. In our realisation, some additional constraints on the selection of pixels are used to reduce further the computation on improper triplet pixels and the risk of false peak formation. Let us use a program segment on pseudo codes to give a clear explanation on the implementation.

\[
A = \left(\frac{1 + (\lambda^2 - 1) \sin^2 \theta}{\lambda}\right)^{1/2}
\]

\[
b = \frac{a}{\lambda}
\]

where

\[
\lambda = \frac{(1 + k) \sin^2 \theta - 1}{(1 + k) \sin^2 \theta - k}
\]

and \( k = A/B \).
Comparison with Tsuji and Matsumoto's model

Before describing the model of Tsuji and Matsumoto, it is interesting to point out that the centre-finding stage of the approach given by Yuen et al. [6] shares some similarity with our centre detection. Referring to Fig. 1, the Hough space in the model of Yuen et al. is updated along the bisector (the dotted line) of the tangents for each edge pair. The common intersections of the bisectors determine the centres of ellipses in the Hough plane. However, this would inevitably generate considerable false votes with the scheme of voting along the line of bisector. For Tsuji and Matsumoto's approach is to formulate a 2-D Hough plane for the representation of the centres of possible candidates and then to extract those edge pixels which have contributed to peak accumulation in an 1-D histogram. By making use of the symmetrical property of the ellipse, the midpoint strategy was applied in Tsuji and Matsumoto's algorithm for the detection of possible centres. To extract the set of possible edge pixels on an elliptical boundary, those pixels which have contributed to peak accumulation in the histogram are determined for the evaluation of parameters with the least square fitting technique. After further calculation and erasing the appropriate group of pixels, the process of curve fitting is repeated for the next elliptical object in the test image. In the case of concentric objects in the test image, it is necessary to iterate the process with the same group of processed pixels for other peaks in the histogram.

Let us compare the algorithms using flow charts as shown in Figs. 3 and 4. The procedures for the estimation of parameters for a candidate ellipse are different between

![Flow chart for the proposed algorithm](image-url)
the two algorithms. Instead of finding five parameters simultaneously for each set of edge pixels, Tsuji and Matsumoto's approach is divided into two parts, namely, (i) gradient estimation and erasing long straight lines, (ii) centre detection by mid-point strategy, (iii) peak detection \((X, Y)\), (iv) extraction of symmetrical edge pair about \((X, Y)\), (v) formation of histogram, (vi) peak detection in the histogram of \(1/R\), (vii) curve fitting.

![Flow chart for Tsuji and Matsumoto's model](image)

Fig. 4  Flow chart for Tsuji and Matsumoto's model

centre detection with midpoint strategy and (ii) curve fitting with a particular group of pixels. As mentioned in the above paragraph, the detection of centres with the midpoint technique is very sensitive to the presence of edge boundaries of other objects. To consider an image like the one in Fig. 5, we may detect multiple local maxima in the corresponding Hough space. Besides the centres of the ellipses, we might still have spurious peaks located in different positions like the centre of a polygon, the midpoint between an elliptical object and a polygon, and even the boundaries of the objects. The formation of peaks nearby the boundaries of the objects is due to the problem of inaccurate gradient estimation. Based on the gradient estimation, \(T_1\) may be defined to have the same gradient as that at \(T_2\). Hence, the midpoint between \(T_1\) and \(T_2\) will be updated in the Hough space according to the midpoint strategy. Similarly, pixels around the boundary of the ellipse contribute many false peaks in corresponding Hough space. In the case of overlapping elliptical objects, the accumulation of these false peaks might easily hazard the performance of the centre detection.

Recently, a modified version of the model of Tsuji and Matsumoto, proposed by Wu and Wang, has been reported [17]. This algorithm restricts the search for symmetrical properties in four specified orientations; the main philosophy of the centre detection, however, is still based on the midpoint strategy for each edge pair. We may appreciate the modified approach as a subset of the conventional approach. Hence, the following discussion will concentrate more often on the model of Tsuji and Matsumoto.

5 Experimental results

For an evaluation of our proposed algorithm, we now provide some of experimental results on both synthetic and real images. The size of the test images was 384 \(\times\) 256 pixels. In the examples that follow, the edge pixels of the objects are extracted from the corresponding gray level images. The extracted elliptical parameters are stored in the Hough domain with the resolution of 2 pixels per interval and \(2\pi\) per interval for the angular one. The total memory requirement for all different Hough planes is 90 Kbytes to store the parameters of the objects.

Fig. 6 shows a synthetic object, in the form as a test ellipse with broken contours. For Tsuji and Matsumoto's algorithm, it is necessary to have a preprocess of erasing the long straight lines in the image so as to prevent the interference from the line pixel. After finding the gradient information and storing into an array according to the corresponding direction, we set a threshold \(\zeta\) for the elimination of long straight lines in the image. The parameter spaces for the proposed algorithm and Tsuji and Matsumoto's algorithm are shown in Figs. 7 and 8, respectively. The actual parameters \((x_c, y_c, a, b, \theta)\) of the synthetic ellipse are \((192, 128, 75, 40, 25^\circ)\). Referring to the data listed in the corresponding Figures, both algorithms achieve very close results. The errors for \((x_c, y_c)\) and \((a, b)\) are about 2 pixels, and are \(-2^\circ\) for \(\theta\). However, the profiles of the respective Hough planes for locating the centres are different from each of others. In Tsuji and
Matsumoto's model, it has been found that multiridges in the profile were detected around the centre of the test ellipse, while the corresponding Hough domain in Fig. 7 for our proposed algorithm gives a noise free result.

Fig. 8 Hough profiles for Tsuji and Matsumoto's model (191, 127, 74, 40, 25°)

To analyse the robustness of the proposed algorithm, we used another test image as shown in Fig. 9 to illustrate the capability of our algorithm with the self-checking property. There are seven synthetic ellipses with different parameters in the test image. Because of the overlapping of a large portion of the boundaries of ellipses, we have detected many spurious peaks in the proposed and Tsuji and Matsumoto's algorithms, respectively, while the Hough profiles are shown in Figs. 12 and 13.

Fig. 11 Detection of image with multiobjects with different parameters: false fitting boundary with a spurious peak in Tsuji and Matsumoto's model

Fig. 12 Hough space for Fig. 9: proposed model

Fig. 13 Hough space for Fig. 9: Tsuji and Matsumoto's model

In Figs. 14-18 we apply both algorithms for the recognition of different objects in real images. Fig. 14 shows the real image captured under a camera, Figs. 15 and 16 show the results of the proposed approach and the results of the Tsuji and Matsumoto's algorithm.
respectively. The corresponding 2-D Hough space for our algorithm and the profile of centre detection with the method of the midpoint extraction are shown in Figs. 17 and 18. A comparison of the Hough profiles of our algorithm along with Tsuji and Matsumoto's model shows that a noisier parameterisation space has been clearly obtained for Tsuji and Matsumoto's approach.

Figs. 19–21 show the results of using our algorithm for the recognition of objects at different conditions. The corresponding back projection of the detected results to the original images are also shown for comparison. In the voting scheme of the conventional Hough techniques, we are faced with the problem of determining the possible values of the threshold. A large threshold may easily lose the vote information which represents an object with small scaling; otherwise, the spurious peaks may be extracted from the Hough space. Furthermore, for the recognition of objects with great difference in scale, it is usually found that the side peaks near the region of the centre of a larger ellipse may be greater than the votes for the centre of a smaller ellipse. As shown in Fig. 21, there

Fig. 15  Verification of proposed model for detection of conic objects in a real image

Fig. 16  Result of Tsuji and Matsumoto's model for detection of conic objects in a real image

Fig. 17  Hough space for Fig. 14: proposed algorithm

Fig. 18  Hough space for Fig. 14: Tsuji and Matsumoto's model

Fig. 19  Testing for the detection of ellipses with different sizes: original image

Fig. 20  Testing for the detection of ellipses with different sizes: detected result

Fig. 21  Histogram of the major axes for corresponding peaks in the centre plane for the image of Fig. 19

is a cup and a coin and the scale between these objects is about five times. We have detected five dominant peaks located in the image, where all three false peaks are indicated by a (+) in the Figure. To identify the false ones among the dominant peaks, a simple search on the Hough space which stored votes for the major axes, the minor axes and $\theta$ can provide a simple solution. Since the possible candidates of the centres have been defined in the Hough plane, the 1-D histogram indicating the length of the major axis for each candidate is shown in Fig. 21. The low value scores and the part with sparse distribution of the histogram can be ignored. By contrast, a sharp peak, which defines the length of the major axis for the corresponding candidate, has been detected in the Hough space. Similarly, we were able to obtain similar distributions in the domains of the minor axis and $\theta$. Hence, for the detection of objects with great differences in scaling, we are able to identify the false peaks from the profiles on the rest of the Hough spaces.

Table 1 compares the detected parameters of the ellipses with the actual data in brackets. The experimen-

tal work was carried out on a 33 MHz 486PC. The average errors of the results are only 1–2 intervals in the Hough domain. In fact, the computation for the recognition is significantly affected by the number of edge pixels in the image. A larger number of edge pixels detected in an image implies a longer processing time spent on the voting for different triplets. The robustness of our algorithm for the detection of elliptical objects under a noisy environment has been tested using many pictures.

The processing time of the proposed algorithm is reported in Table 2. The corresponding data for Tsuji and Matsumoto’s approach are also shown for a comparison. However, for test images like Fig. 19, we have not reported Tsuji and Matsumoto’s processing time because of the false detection. According to Table 2, the computation speed for Tsuji and Matsumoto’s model is faster than our proposed algorithm. The difference can be explained by the modeling of the voting scheme. The complexity of our approach is in the order of $N_C$ instead of $N_C^2$ for Tsuji and Matsumoto’s model. Hence, the number of possible voting instance is larger in our algorithm. Furthermore, because of the elimination of pixels lying on straight lines, the number of edge pixel $N'$ for Tsuji and Matsumoto’s approach is smaller than that of the new algorithm, $N$. An increase in the processing time is the price to be paid for a significant enhancement the performance of the recognition rate. This also forms a fruitful direction for further research.

6 Conclusion

We have proposed a new scheme for ellipse detection which is different from other approaches to be found in the literature. The present algorithm provides a fast and effective verification for each voting in the corresponding Hough space. This property is inherent in our algorithm. Instead of random sampling or partitioning the image into many subimages, some useful relationships of the conic trigonometry are applied to reduce the computation for the recognition. Compared with the conventional approach, a significant improvement is obtained using this novel approach. Because of the self-verification feature in the suggested technique, it is very useful for the detection of real images with an environment having mixed objects.

In connection with the memory requirement, the conventional 5-D Hough space is replaced by four 2-D Hough planes. In our experiment, we have used 90 Kbytes of memory for the storage of the Hough planes. In fact, at the cost of a slight decrease on the accuracy of the recognition, the demand on the memory space can be greatly reduced, to about 28 Kbytes by a decrease of the resolution to 4 pixels width per interval for spatial domain and $2^\circ$ per interval for orientations.

Using our suggested algorithm for the recognition of circular objects, we are able to use an even smaller Hough space to store the parameters of the patterns. The planes for the minor axis and the relative orientation can be eliminated for detection purposes. However, to cater for the situation of the presence of both types of objects, we have to use four 2-D planes for recognition purposes.

7 References