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On the uniqueness of Gaussian ansatz parameters equations: generalized projection operator method

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Abstract

Based on the recently reported generalized projection operator method for nonlinear Schrödinger equation, one can derive two different sets of pulse parameters equations while using ansätze like hyperbolic secant or raised-cosine. We show that in case of a Gaussian like ansatz those sets of equations are unique because of the symmetric property between the ansatz parameters.

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Almost all the complex nonlinear partial differential equations (NPDEs) governing the nonlinear systems are in the family of nonlinear Schrödinger equation (NLSE). Many important physical systems like, nonlinear fiber optics, Bose–Einstein condensate (BEC), water waves, plasma waves, etc., are governed by the NLSE [1]. The NLSE is completely integrable and its N-soliton solutions can be obtained using the standard inverse scattering transform scheme [2]. The completely integrable form of NLSE is a very special case or approximated version of the equation governing real systems. Many numerical, approximation and perturbation methods were developed to study the pulse dynamics of such nonintegrable NLSE. Lagrangian variational method (LVM) is one of the famous approximation technique often used in nonlinear fiber optics, fluid dynamics and in BEC for investigating the pulse dynamics of the systems governing NPDEs [3,4].

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In 1988, Boesch et al. [5] proposed a projection operator scheme for the Klein–Gordon equation which can preserve the Hamiltonian structure of the dynamical equation. Very recently we have reported that Boesch et al., projection operator scheme cannot be straightforwardly applied to complex NPDEs like NLSE [6]. Rather we showed that a generalized projection operator scheme must be used for deriving the pulse dynamical equations for NLSE. The generalized projection operator contains a phase constant parameter say, θ whose value set to 0 or $\pi/2$ will determine whether we have minimized the Lagrangian corresponding to the NLSE or the residual field which is the difference between the exact pulse and the assumed ansatz function. Minimizing the Lagrangian of the NLSE is equivalent to the famous LVM [3] and minimizing the residual field is equivalent to the bare approximation (BA) of the collective-variable (CV) theory [7]. In general an ansatz function will result two sets of ordinary differential equations (ODEs) from the generalized projection operator method each one corresponds to the LVM and the BA of the CV method, respectively. In the same work we showed that the hyperbolic secant ansatz and raised cosine ansatz give different sets of ODEs for the NLSE and which set of ODEs best approximates the pulse parameter dynamics depends on the ansatz used. Tchofo Dinda et al., have chosen a Gaussian ansatz as an example for the BA of the CV method, which has produced the same CV equations of motions from both the methods [7]. In this Letter we show that the Gaussian like ansatz has its own symmetry relations between the CVs which makes the final ODEs from both the methods the same.

We consider the NLSE in the form:

$$\psi_z + \frac{i\beta}{2}\psi_{tt} - i\gamma|\psi|^2\psi = 0, \tag{1}$$

where ψ is the slowly varying envelope of the axial electrical field, β and γ represent the group-velocity dispersion and self-phase modulation parameters, respectively. These definitions of the variables and parameters are for the context of envelope soliton propagation in optical fibers. These definitions will vary in other physical systems governed by the NLSE [1]. Let us introduce the ansatz function as $f(x_1,\dots,x_N,t)$, where $x_1,\dots,x_N$ are the pulse parameters (also called CVs) dependent only on z. For complex equation like the NLSE let us introduce a generalized projection operator $\mathcal{P}_k = \exp(i\theta) f^*_{x_k}$, where θ is an arbitrary phase constant. To obtain the CVs equations of motion we project Eq. (1) in the direction of $\mathcal{P}_k$. We substitute the ansatz function f for ψ in Eq. (1), multiply the resulting equation by $\mathcal{P}_k$ and integrate with respect to t we obtain

$$\int_{-\infty}^{\infty} \Re\left[f_z f^*_{x_k}\exp(i\theta)\right]dt - \frac{\beta}{2}\int_{-\infty}^{\infty} \Im\left[f_{tt} f^*_{x_k}\exp(i\theta)\right]dt + \gamma\int_{-\infty}^{\infty} |f|^2\Im\left[f f^*_{x_k}\exp(i\theta)\right]dt = 0. \tag{2}$$

If we substitute $\theta = \pi/2$ in Eq. (2), we get the equivalent Lagrangian variations [6]

$$\int_{-\infty}^{\infty} \Im\left[f_z f^*_{x_k}\right]dt - \frac{\beta}{2}\int_{-\infty}^{\infty} \Re\left[f_{tt} f^*_{x_k}\right]dt + \gamma\int_{-\infty}^{\infty} |f|^2\Re\left[f f^*_{x_k}\right]dt = 0. \tag{3}$$

If we substitute $\theta = 0$ in Eq. (2), we get the minimization of residual field which is equivalent to the BA of the CV theory [6]

$$\int_{-\infty}^{\infty} \Re\left[f_z f^*_{x_k}\right]dt - \frac{\beta}{2}\int_{-\infty}^{\infty} \Im\left[f_{tt} f^*_{x_k}\right]dt + \gamma\int_{-\infty}^{\infty} |f|^2\Im\left[f f^*_{x_k}\right]dt = 0. \tag{4}$$

Now, let us see how the final CVs equations of motions derived from the LVM or the BA of the CV theory for a standard Gaussian ansatz with four CVs are unique. Consider the Gaussian ansatz as

$$f = x_1\exp\left(\frac{-t^2}{x_2^2} + \frac{i x_3 t^2}{2} + i x_4\right), \tag{5}$$

where $x_1,\ x_2,\ x_3/(2\pi)$ and x_4 represent the pulse amplitude, width, chirp and phase, respectively. Using Eq. (5) in (3) and (4), we get the same CV equations of motions as

$$\dot{x}_1 = \frac{\beta x_1 x_3}{2}, \tag{6a}$$

$$\dot{x}_2 = -\beta x_2 x_3, \tag{6b}$$

$$\dot{x}_3 = \beta\left(x_3^2 - \frac{4}{x_2^4}\right) - \frac{\sqrt{2}\gamma x_1^2}{x_2^2}, \tag{6c}$$

$$\dot{x}_4 = \frac{\beta}{x_2^2} + \frac{5\gamma x_1^2}{4\sqrt{2}}. \tag{6d}$$

When we substitute the Gaussian ansatz (5) either in Eqs. (3) or (4) and before deriving the final simplified form of the ODEs as Eqs. (6), we get the matrix equation (note that $\theta = \pi/2$ for Eq. (3) and $\theta = 0$ for Eq. (4))

$$[M_\theta][\dot{x}] = [F_\theta], \tag{7}$$

where

$$[x] \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad [F_\theta] \equiv \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}, \qquad [M_\theta] \equiv \begin{bmatrix} \frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} & \frac{\partial M_1}{\partial x_3} & \frac{\partial M_1}{\partial x_4} \\ \frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2} & \frac{\partial M_2}{\partial x_3} & \frac{\partial M_2}{\partial x_4} \\ \frac{\partial M_3}{\partial x_1} & \frac{\partial M_3}{\partial x_2} & \frac{\partial M_3}{\partial x_3} & \frac{\partial M_3}{\partial x_4} \\ \frac{\partial M_4}{\partial x_1} & \frac{\partial M_4}{\partial x_2} & \frac{\partial M_4}{\partial x_3} & \frac{\partial M_4}{\partial x_4} \end{bmatrix} \tag{8}$$

with

$$\frac{\partial M_j}{\partial x_k} = \int_{-\infty}^{\infty} \Re\big[f_{x_j} f_{x_k}^* \exp(i\theta)\big]\,dt,$$

$$F_k = \frac{\beta}{2}\int_{-\infty}^{\infty} \Im\big[f_{tt} f_{x_k}^* \exp(i\theta)\big]\,dt - \gamma\int_{-\infty}^{\infty} |f|^2 \Im\big[f f_{x_k}^* \exp(i\theta)\big]\,dt.$$

For $\theta = 0$ or $\theta = \pi/2$, we get different sets of Eq. (7), for the case of Gaussian ansatz (5). However, after solving those simultaneous equations we get the same set of ODEs (6) because of the availability of the transformation:

$$[R][M_{\theta=\pi/2}] = [M_{\theta=0}], \tag{9a}$$

$$[R][F_{\theta=\pi/2}] = [F_{\theta=0}], \tag{9b}$$

where

$$[R] = \begin{bmatrix} 0 & 0 & 0 & 1/x_1 \\ 0 & 0 & 4/x_2^3 & 0 \\ 0 & -x_2^3/4 & 0 & 0 \\ -x_1 & 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

The transformation in (9) is available for the Gaussian ansatz as a result of the symmetry property between the CV's in the Gaussian ansatz function. We find that the derivatives of the Gaussian ansatz function (5) with respect to the CV's (f_{x_k}) are related to other f_{x_k}, i.e., $f_{x_1} = -iR_{14} f_{x_4}$, $f_{x_2} = -iR_{23} f_{x_3}$, $f_{x_3} = -iR_{32} f_{x_2}$ and $f_{x_4} = -iR_{41} f_{x_1}$, where R_{uv} represents the elements of the matrix $[R]$ given by Eq. (10). This special property of the Gaussian ansatz makes the two different Eq. (7), for the values of θ equal to 0 or $\pi/2$ to result in the same set of ODEs (6).

So far we have discussed about the Gaussian ansatz function (5) with four CVs, which resulted same set of ODEs from LVM and the BA of the CV theory. We have also proved that the availability of the transformation with simple matrix (10) which connects the derivatives of the Gaussian ansatz function is the reason for the uniqueness of the ODEs resulting from the generalized projection operator method. Now let us consider the more general Gaussian ansatz with six pulse parameters as

$$f = x_1 \exp\left[\frac{-(t-x_c)^2}{x_2^2} + \frac{i x_3 (t-x_c)^2}{2} + i x_f (t-x_c) + i x_4\right], \tag{11}$$

where x_c and x_f represent the pulse temporal position and frequency, respectively. Using Eq. (11) in (3) and (4), we get the same CV equations of motions as (6) and two more equations for pulse temporal position and frequency as

$$\dot{x}_c = -\beta x_f, \tag{12a}$$

$$\dot{x}_f = 0. \tag{12b}$$

Interesting feature about the Gaussian ansatz with six CVs is that like the four pulse parameter case, this one also have similar transformation like Eq. (9) but with the matrix R given as

$$[R] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{x_1} \\ -x_1 x_f & \frac{x_2^2 x_3}{2} & 0 & 0 & \frac{-4 - x_2^4 x_3^2}{2x_2^2} & \frac{-x_2^2 x_3 x_f}{2} \\ 0 & 0 & 0 & \frac{4}{x_2^3} & 0 & 0 \\ 0 & 0 & \frac{-x_2^3}{4} & 0 & 0 & 0 \\ 0 & \frac{x_2^2}{2} & 0 & 0 & \frac{-x_2^2 x_3}{2} & \frac{-x_2^2 x_f}{2} \\ -x_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

Thus transformation (9) is still available for the Gaussian ansatz (11) as a result of the symmetry property between the CV's are still preserved after the introduction of two other pulse parameters namely the pulse temporal position (x_c) and frequency (x_f). Here also we find that the derivatives of the Gaussian ansatz function (11) with respect to the CV's (f_{x_k}) are related to other f_{x_k} in a more complicated form, i.e., $f_{x_1} = -iR_{16} f_{x_4}$, $f_{x_c} = -iR_{21} f_{x_1} - iR_{22} f_{x_c} - iR_{25} f_{x_f} - iR_{26} f_{x_4}$, $f_{x_2} = -iR_{34} f_{x_3}$, $f_{x_3} = -iR_{43} f_{x_2}$, $f_{x_f} = -iR_{52} f_{x_c} - iR_{55} f_{x_f} - iR_{56} f_{x_4}$ and $f_{x_4} = -iR_{61} f_{x_1}$, where R_{uv} represents the elements of the matrix $[R]$ given by Eq. (13). This special property of the Gaussian ansatz makes the two different Eq. (7), for the values of θ equal to 0 or $\pi/2$ to result in the same set of ODEs (6) and (12). This is the reason why the example of Gaussian ansatz lead Tchofo Dinda et al., to prematurely conclude in Ref. [7] that the LVM and the BA of the CV theory to be the same.

To conclude, in this Letter we have proved for ansätze like Gaussian which has an inherent symmetric property between the CVs will result the same set of the dynamical equations derived either from the LVM or from the BA of the CV theory. Hence while studying the dynamics of NLSE governed system with variational analysis using a Gaussian ansatz then there is no need to separately investigate them for LVM and the BA of the CV theory. But for other ansätze like hyperbolic secant or raised cosine functions, which lacks this kind of symmetric property between the pulse parameters, one needs to derive both sets of ODEs and analyze their dynamics.

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References

[1] L. Vázquez, L. Streit, V.M. Pérez-García (Eds.), Nonlinear Schrödinger and Klein–Gordon Systems: Theory and Applications, World Scientific, Singapore, 1996.
[2] V.E. Zakharov, A.B. Shabat, Sov. Phys. JETP 23 (1972) 142.
[3] D. Anderson, Phys. Rev. A 27 (1983) 3135.
[4] B.A. Malomed, Prog. Opt. 43 (2003) 71.
[5] R. Boesch, P. Stancioff, C.R. Willis, Phys. Rev. B 38 (1988) 6713.
[6] K. Nakkeeran, P.K.A. Wai, Generalized projection operator method to derive the pulse parameters equations for the nonlinear Schrödinger equation, Opt. Commun., in press.
[7] P. Tchofo Dinda, A.B. Moubissi, K. Nakkeeran, Phys. Rev. E 63 (2001) 016608.