



Generalized projection operator method to derive the pulse parameters equations for the nonlinear Schrödinger equation

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Abstract

We present a novel projection operator method for deriving the ordinary differential equations (ODEs) which describe the pulse parameters dynamics of an ansatz function for the nonlinear Schrödinger equation. In general, each choice of the phase factor θ in the projection operator gives a different set of ODEs. For $\theta = 0$ or $\pi/2$, we prove that the corresponding projection operator scheme is equivalent to the Lagrangian method or the bare approximation of the collective variable theory. Which set of ODEs best approximates the pulse parameter dynamics depends on the ansatz used.

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1. Introduction

Many optical systems such as optical fiber communication systems, planar waveguides and other guided wave transmission systems are generally governed by the family of nonlinear Schrödinger

equation (NLSE). Fundamental NLSE in optical fiber system includes only the group-velocity dispersion and Kerr nonlinearity terms. Other equations in the family of NLSE are usually called as generalized NLSE or higher-order NLSE, which may include other effects like optical losses, higher-order dispersion, stimulated inelastic scattering and self-steepening effects. Under special cases, the NLSE is completely integrable and the corresponding soliton solutions can be derived using

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the standard technique called inverse scattering transform. But the family of NLSE equations governing most practical cases like conventional fiber transmission system, dispersion-managed (DM) fiber system are not completely integrable in general. Even though, some perturbation methods were reported to investigate the behavior of physically interesting non-integrable NLSE family, researchers working in nonlinear optics mostly rely on numerical methods and Lagrangian variational method (LVM) to study the system dynamics [1,2]. LVM is one of the widely used approximation techniques which has been applied to study the dynamics of various pulse parameters with respect to the system parameters, to estimate pulse interaction length and to find the fixed point solutions of DM fiber systems [2]. There are numerous works related to the modification of the LVM to include various important effects neglected in the formalism of typical variational method. One of the important factors considered by various researcher to modify the LVM has been on the radiation induced solitons interactions [3–5]. Kuznetsov et al. [6] considered the nonlinear interaction of solitons and radiations, where they reported the unsuccessful application of the variational method. Nevertheless all these works were based on the fundamental LVM and its success or failure. Mikhailov [7] has reported an interesting article about the validity of the LVM.

Boesch et al. [8] proposed a new projection operator scheme for the Klein–Gordon equation to derive the ordinary differential equations (ODEs) which describe the pulse dynamics of the Klein–Gordon equation. Tchofo Dinda et al. [9] proposed a collective variable (CV) theory which was claimed to be equivalent to the LVM under the umbrella of bare approximation (BA). Later in the paper, we discuss more in detail the idea behind the CV theory and its limit of BA. Here, we present a generalized projection operator method for deriving the ansätze parameters ODEs of the NLSE. For the choice of 0 or $\pi/2$ to the phase constant in the projection operator, we prove that the corresponding projection operator scheme is equivalent to the LVM or the BA of the CV theory. We show that the two methods in general give two different sets of ODEs for the ansätze para-

meter of the NLSE. We consider the NLSE in the form

$$\psi_z + \frac{i\beta}{2}\psi_{tt} - i\gamma|\psi|^2\psi = 0, \quad (1)$$

where ψ is the slowly varying envelope of the axial electrical field, β and γ represent the group-velocity dispersion and self-phase modulation parameters, respectively. The NLSE (1) is completely integrable for constant values of β and γ . In case of systems like DM fiber transmission line, the parameters β and γ will become z dependent which would result in a non-integrable NLSE. Thus, it becomes important to derive the ODE dynamical equations of the NLSE (1). Let us introduce the ansatz function as $f(x_1, \dots, x_N, t)$, where x_1, \dots, x_N are the pulse parameters (also called CVs). The projection operator scheme developed in [8] is for a real equation like Klein–Gordon equation. If we extend that projection operator scheme to a complex equation like NLSE directly, then the resulting number of ODEs will be twice the number of the CVs used because both the real and imaginary parts after the projection must be set to zero. The main idea behind the introduction of the projection operator scheme [8] is to preserve the Hamiltonian structure and indirectly the Lagrangian structure. As per the present form of the NLSE in Eq. (1), if we want to preserve the Lagrangian structure while applying the projection operator scheme we must set the imaginary part of Eq. (1) after projection to zero only and ignore the real part. This is because only setting the imaginary part to zero will guarantee the retaining of the term $(i/2)(f^*f_z - ff_z^*)$ in the Lagrangian of the NLSE. On the other hand, if we write NLSE in a different form as $i \times$ (Eq. (1)) then apply the projection operator scheme, we need to set the real part after projection to zero to retain the same term $(i/2)(f^*f_z - ff_z^*)$. Thus, one cannot directly extend the projection operator scheme to a complex equation like the NLSE. To avoid this inconsistency in applying the Boesch et al. [8] projection operator scheme to a complex equation like the NLSE we introduce a generalized projection operator $\mathcal{P}_k = \exp(i\theta)f_{x_k}^*$, where θ is an arbitrary phase constant. To obtain the CVs equations of motion we project Eq. (1) in the direction of \mathcal{P}_k . We sub-

stitute the ansatz function f for ψ in Eq. (1), multiply the resulting equation by \mathcal{P}_k , integrate with respect to t and take the real part, we obtain

$$\int_{-\infty}^{\infty} \Re [f_z f_{x_k}^* \exp(i\theta)] dt - \frac{\beta}{2} \int_{-\infty}^{\infty} \Im [f_u f_{x_k}^* \exp(i\theta)] dt + \gamma \int_{-\infty}^{\infty} |f|^2 \Im [f f_{x_k}^* \exp(i\theta)] dt = 0. \tag{2}$$

In deriving Eq. (2) we considered the real part. One can also consider the imaginary part as it is equivalent to Eq. (2) with a simple phase change $\theta \rightarrow \theta + \pi/2$. The same explanation holds if one considers the different form of the NLSE as $i \times$ (Eq. (1)). If we substitute $\theta = \pi/2$ in Eq. (2), we get

$$\int_{-\infty}^{\infty} \Im [f_z f_{x_k}^*] dt - \frac{\beta}{2} \int_{-\infty}^{\infty} \Re [f_u f_{x_k}^*] dt + \gamma \int_{-\infty}^{\infty} |f|^2 \Re [f f_{x_k}^*] dt = 0. \tag{3}$$

Eq. (3) is equivalent to the variations

$$\frac{\partial L}{\partial x_k} - \frac{d}{dz} \left(\frac{\partial L}{\partial \dot{x}_k} \right) = 0, \tag{4}$$

(where the over-head dot represents the derivative with respect to z) on the NLSE Lagrangian

$$L = \int_{-\infty}^{\infty} \left[\frac{\beta}{2} |f|^2 + \frac{\gamma}{2} |f|^4 + \frac{i}{2} (f^* f_z - f f_z^*) \right] dt. \tag{5}$$

There is a change in the order of computation between Eqs. (3) and (4). In Eq. (3), the variations are followed by the integration. Whereas in Eq. (4) the integration is followed by the variations. This proves the equivalence between the projection operator scheme, which preserves the Hamiltonian structure proposed in [8] and the LVM.

Tchofo Dinda et al. [9] proposed the CV theory for the optical solitons. Using the Gaussian ansatz as an example they concluded that the BA in minimizing the residual-field energy is equivalent to the LVM. But fundamentally these two methods are different. The residual-field energy is defined as

$$\epsilon = \int_{-\infty}^{\infty} |q|^2 dt = \int_{-\infty}^{\infty} |\psi - f|^2 dt. \tag{6}$$

To minimize this residual-field energy both in the beginning of the optical fiber and also during the evolution, a couple of constraints has been introduced [9]. Applying those constraints and also from the BA ($q \rightarrow 0$), we get

$$\frac{d}{dz} \left(\frac{d\epsilon}{dx_k} \right) = \int_{-\infty}^{\infty} \Re [f_z f_{x_k}^*] dt - \frac{\beta}{2} \int_{-\infty}^{\infty} \Im [f_u f_{x_k}^*] dt + \gamma \int_{-\infty}^{\infty} |f|^2 \Im [f f_{x_k}^*] dt = 0. \tag{7}$$

To minimize the residual-field energy, we need to collect the real part of $f_z f_{x_k}^*$, which can be done by substituting $\theta = 0$ in Eq. (2). Hence the BA of the CV theory Eq. (2) becomes

$$\int_{-\infty}^{\infty} \Re [f_z f_{x_k}^*] dt - \frac{\beta}{2} \int_{-\infty}^{\infty} \Im [f_u f_{x_k}^*] dt + \gamma \int_{-\infty}^{\infty} |f|^2 \Im [f f_{x_k}^*] dt = 0. \tag{8}$$

We further note that Eq. (2) can be rewritten as (Eq. (8)) $\times \cos \theta -$ (Eq. (3)) $\times \sin \theta = 0$ which shows that our projection operator scheme for any other values of θ apart from 0 or $\pi/2$ is equivalent to the linear combination of the LVM and the BA of the CV theory. Now, let us see how the final CVs equations of motions derived from the LVM or the BA of the CV theory for a standard ansatz like hyperbolic secant are different. Consider a hyperbolic secant ansatz in the form

$$f = x_1 \operatorname{sech} \left(\frac{t}{x_2} \right) \exp \left(\frac{ix_3 t^2}{2} + ix_4 \right), \tag{9}$$

where $x_1, x_2, x_3/(2\pi)$ and x_4 represent the pulse amplitude, width, chirp and phase, respectively. Using Eq. (9) in (3) and (8), we get the same functional form for the CV equations of motions as

$$\begin{aligned} \dot{x}_1 &= \frac{\beta x_1 x_3}{2}, & \dot{x}_2 &= -\beta x_2 x_3, \\ \dot{x}_3 &= \beta \left(x_3^2 - \frac{\alpha_1}{x_2^4} \right) - \frac{\alpha_2 \gamma x_1^2}{x_2^2}, \\ \dot{x}_4 &= \frac{\alpha_3 \beta}{x_2^2} + \alpha_4 \gamma x_1^2 \end{aligned} \tag{10}$$

but the constants α_n 's ($n = 1-4$) are different. For Eq. (3), the α_n 's are given by

$$\alpha_1 = \frac{4}{\pi^2}, \quad \alpha_2 = \frac{4}{\pi^2}, \quad \alpha_3 = \frac{1}{3}, \quad \alpha_4 = \frac{5}{6} \quad (11)$$

whereas the constants α_n 's for Eq. (8) are given by

$$\alpha_1 = \frac{30}{\pi^4}, \quad \alpha_2 = \frac{30}{\pi^4}, \quad \alpha_3 = \frac{1}{6} + \frac{5}{4\pi^2}, \quad (12)$$

$$\alpha_4 = \frac{2}{3} + \frac{5}{4\pi^2}.$$

Thus, the LVM and the BA of the CV theory give different sets of ODEs for the same sech ansatz function.

As there is only one set of ODEs from both the methods for the case of the Gaussian ansatz,

there is no question about which method better approximates the dynamics of the NLSE. For ansätze that do not possess such symmetry there will always be a question about which ODEs resulted from the two different methods can better approximate the correct dynamics. To answer this question in case of the sech ansatz, we consider the pulse propagation in a typical DM fiber transmission line, with a periodic dispersion management using two types of fibers: dispersions ± 12.5 ps/nm/km (β), for both types of fibers the nonlinear coefficient is taken as $0.002 \text{ W}^{-1}\text{m}^{-1}$ (γ) and the anomalous and normal dispersion fiber lengths are 21 and 20.2 km, respectively. Input pulse is the fixed point solution (for solving the NLSE numerically) of the DM fiber system

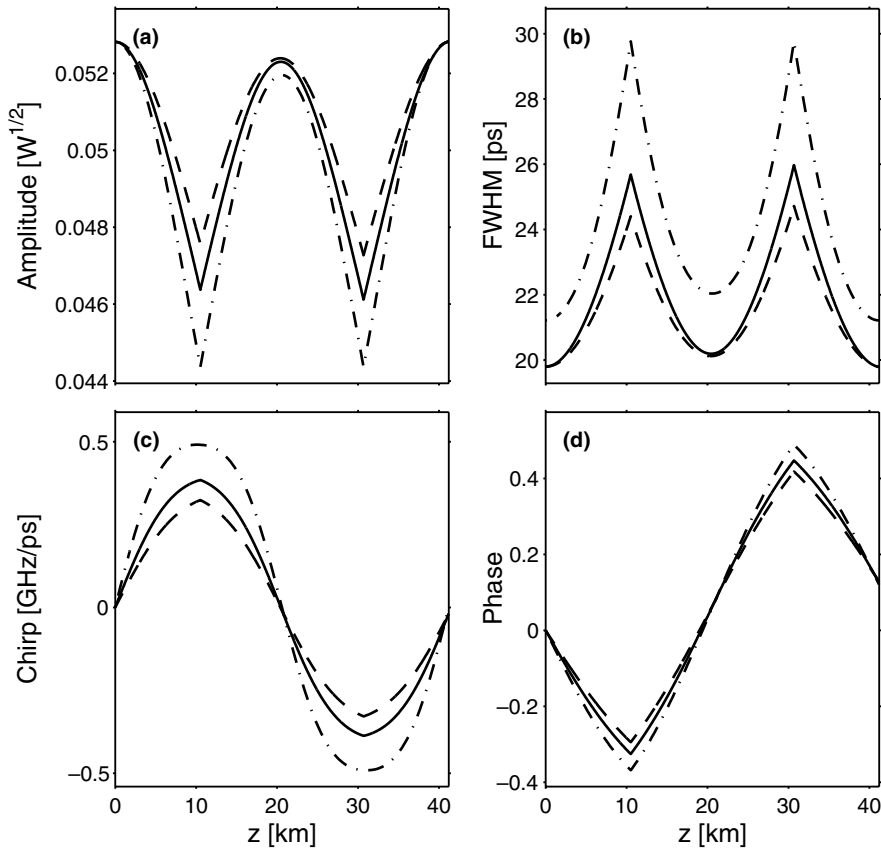


Fig. 1. Evolution of the pulse (a) amplitude, (b) width, (c) chirp and (d) phase in one dispersion map. Solid, dashed and dot-dashed curves show the solutions of the sech ansatz pulse dynamical equations (10) with (11), sech ansatz equations (10) with (12) and the NLSE (1), respectively.

with FWHM 21.2 ps and energy 0.063 pJ (for solving the ODEs derived from both the LVM and the BA of the CV theory). Fig. 1 shows the comparison between the numerical solution of the NLSE (dot-dashed curves) and the sech ansatz ODEs solution (solid and dashed curves represents the LVM and the BA of the CV theory, respectively). Fig. 1(a)–(d) show the evolution of the pulse amplitude, width, chirp and phase in one dispersion map, respectively. Note that while fixing the initial conditions for different approaches it is not possible to have same pulse profile, energy, amplitude and width, simultaneously. Hence, we have fixed the energy and the amplitude to be the same for all the three cases and while fixing the relevant pulse shape (sech);

the pulse width becomes different from the numerical DM soliton pulse. This is the reason why the initial pulse widths for all the three cases are not same in Fig. 1(b). In the case of the sech ansatz, it is obvious that the ODEs resulted from the LVM is better than the BA of the CV theory in predicting the correct dynamics of the NLSE in DM systems.

Now we consider a different ansatz function, namely the raised-cosine (RC) ansatz. This ansatz will be helpful to study the dynamics of the pulses in optical fibers, which are originated from Mach-Zender modulator. More detailed study on the dynamics of RC ansatz pulse propagation in DM fiber system has been reported in [10]. The RC ansatz takes the form

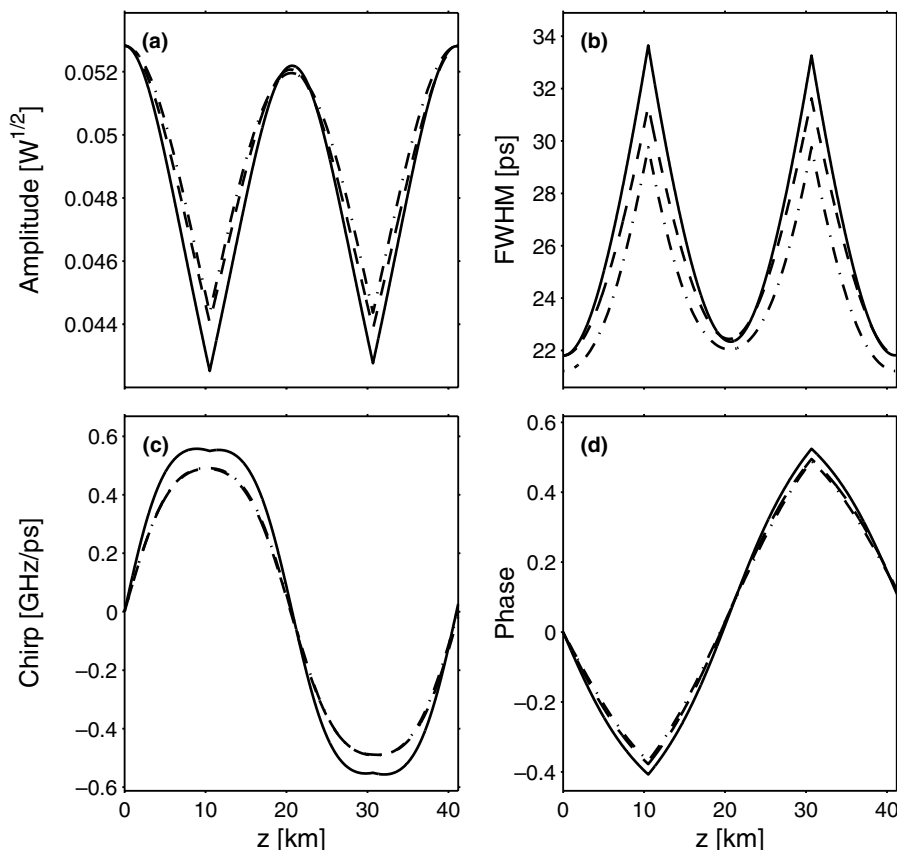


Fig. 2. Evolution of the pulse (a) amplitude, (b) width, (c) chirp and (d) phase in one dispersion map. Solid, dashed and dot-dashed curves show the solutions of the RC ansatz pulse dynamical equations (10) with (14), RC ansatz equations (10) with (15) and the NLSE (1), respectively.

$$f = \frac{x_1}{2} \left[1 + \cos \left(\frac{\pi t}{x_2} \right) \right] \exp \left(\frac{ix_3 t^2}{2} + ix_4 \right). \quad (13)$$

Using Eq. (13) in (3) and (8), we get the same functional form for the CV equations of motions as in the case of sech ansatz as Eq. (10). For the LVM from Eq. (3), the α_n 's are given by (integration is carried out between the limits $-x_2$ and x_2 which will include only one pulse for the RC ansatz)

$$\begin{aligned} \alpha_1 &= \frac{2\pi^4}{2\pi^2 - 15}, & \alpha_2 &= \frac{35\pi^2}{16(2\pi^2 - 15)}, \\ \alpha_3 &= \frac{\pi^2}{3}, & \alpha_4 &= \frac{175}{192}, \end{aligned} \quad (14)$$

whereas the constants α_n 's for the BA of the CV theory from Eq. (8) are given by

$$\begin{aligned} \alpha_1 &= \frac{60\pi^4}{16\pi^4 - 600\pi^2 + 4545}, \\ \alpha_2 &= \frac{4375\pi^2}{48(16\pi^4 - 600\pi^2 + 4545)}, \\ \alpha_3 &= \frac{\pi^2(16\pi^4 - 540\pi^2 + 4095)}{6(16\pi^4 - 600\pi^2 + 4545)}, \\ \alpha_4 &= \frac{35(192\pi^4 - 6950\pi^2 + 52665)}{576(16\pi^4 - 600\pi^2 + 4545)}. \end{aligned} \quad (15)$$

Fig. 2 shows the comparison between the numerical solution of the NLSE (dot-dashed curves) and the RC ansatz ODEs solution (solid and dashed curves represents the LVM and the BA of the CV theory, respectively) for the same DM fiber system considered in Fig. 1. In contrast to the sech ansatz, the solution of the ODEs derived from the BA of the CV theory for the RC ansatz is closer to the exact numerical solution than the LVM.

To conclude, in this paper using a novel projection operator scheme for a choice of θ equal to 0 or $\pi/2$, we have established that with respect to the assumed ansätze, the NLSE will have either the same set of dynamical equations (Gaussian ansatz) or two different sets of dynamical equations (sech

and RC ansätze). These two different choices of the projection operator scheme is found to be equivalent to the LVM and the BA of the CV theory. For ansätze different from Gaussian, the set of the dynamical equations derived either from the LVM (sech ansatz) or from the BA of the CV theory (RC ansatz) will better approximate the correct dynamics of the NLSE in case of the DM fiber system considered in this work. Hence, we stress that for any ansatz function, one needs to study both sets of ODEs corresponding to the LVM and the BA of the CV theory when one applies the variational analysis on any complex equations like the NLSE.

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