Reconstruction from 2-D wavelet transform modulus maxima using projection

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Abstract: Wavelet transform modulus maxima can be used to characterise sharp variations such as edges and contours in an image. The authors analyse the a priori constraints present in the wavelet transform modulus maxima representation. A new projection-based algorithm which enforces all the a priori constraints in the representation is proposed. Quadratic programming is used to obtain a sequence which satisfies the maxima constraint, thus realising the projection onto the maxima constraint space. To save computation, an approximate method to obtain a sequence which satisfies the maxima constraint is given. The new algorithm is shown to provide better solution than the original reconstruction algorithm of Mallat and Zhong. The authors also propose a simple method to accelerate the algorithm. The acceleration is achieved by the incorporation of a momentum term which exploits the high correlation between the difference images between two consecutive iterations. The simulation results show that the proposed algorithm gives good reconstruction and the simple acceleration method can significantly improve the convergence rate.

1 Introduction

Points of sharp variation such as edges and discontinuities are usually one of the most important features for analysing the properties of signals or images. However, since these features usually exist in a range of scale, a multiscale strategy needs to be adopted. The need to adopt a multiscale strategy for describing a signal or image is also brought out by the study on the biological visual system. It was conjectured that the basic representation, the primal raw sketch, furnished by the retinal system is a succession of contour sketches at scales that are in geometric progression [1]. The wavelet transform modulus maxima representation (WTMM) proposed by Mallat and Zhong [2] provides such a multiscale contour representation of an image. This representation is obtained by retaining the local modulus maxima of the continuous dyadic wavelet transform (WT) which correspond to discontinuities in an image. It provides a compact but physically meaningful description of an image. Two important issues related to the WTMM are: first, 1. the uniqueness of the representation; and secondly, 2. the reconstruction of an image from its WTMM. The uniqueness issue of the representation has been investigated for 1-D signal [3, 4]. The uniqueness of the representation depends on the total number of WT modulus maxima in the signal. If the maxima points are sparse, the representation is, in general, not unique. In that situation, it will still be desirable if a close approximation of the original signal can be reconstructed.

A number of reconstruction methods have been proposed for the 1-D WTMM reconstruction problem such as the projection-based method [2, 5], the conjugate gradient error minimisation method [6] and the least square eigenspace method [7]. For the 2-D WTMM reconstruction problem, Mallat and Zhong [2, 8] proposed a projection onto convex space method. In this method, the a priori constraints are formulated as some convex sets. An alternating projection technique is then used to obtain a feasible solution at the intersection of all the convex sets. As one of the a priori constraint in the WTMM is not convex, their algorithm does not enforce the constraint strictly [4]. We will show that this affects the quality of the reconstruction.

In this paper, we propose a new projection algorithm which enforces the a priori constraint strictly in an efficient way. For any iterative algorithm, a main concern is the convergence rate. We propose a way to improve the convergence rate based on the idea of momentum. Some experimental results will be presented to show the performance of our algorithm.

2 The reconstruction problem

Let the wavelet functions $\Psi_1(x, y)$ and $\Psi_2(x, y)$ be the partial derivatives of a 2-D smoothing function, i.e. a cubic spline [2, 8]. Thus the 2-D dyadic WT of an image $f(x, y)$ becomes the gradient of the image at multiples of dyadic scales. The two components of the 2-D dyadic WT are denoted as $W_{2j}^1 f(x, y)$ and $W_{2j}^2 f(x, y)$. The local modulus maxima of $W_{2j}^1 f(x, y)$ along $x$, for constant $y$, are the points of sharper horizontal variation of $f(x, y)$ smoothed at the scale $2^j$. Similarly, the local modulus maxima of $W_{2j}^2 f(x, y)$ along $y$, for constant $x$, are the points of sharper vertical variation. The local maxima belong to curves in the $(x, y)$ plane which are the edges of the image along each
direction [2, 8]. Let \( X^j \) be the set of modulus maxima locations of \( W^j_2(f(x, y)) \) along the horizontal direction, i.e.

\[
X^j = \{(x_j, y) : |W^j_2(f(x, y))| \text{ has local maxima at (x, y) along the horizontal direction}\}
\]

(1)
The collection of \( X^j \) for all scales \( j \) is denoted by \( X_1 = \{X^j\}_{j \in \mathbb{Z}} \). Similarly

\[
X_2^j = \{(x, y_j) : |W^j_2(f(x, y))| \text{ has local maxima at (x, y) along the vertical direction}\}
\]

(2)
and \( X_2 = \{X^j\}_{j \in \mathbb{Z}} \). For a \( J \)-level 2-D dyadic WT, the collection

\[
\{(W^j_2(x_j, y), W^j_2(x, y_j), S_2^j(f(x, y))\}_{1 \leq j \leq J} \}
\]

(3)
where \( 1 \leq j \leq J \), \((x_j, y_j) \in X_1 \), \((x, y_j) \in X_2 \), is called the 2-D WTMM of \( f(x, y) \). \( S_2^j(f(x, y)) \) is the lowpass approximation at level \( J \).

The 2-D WTMM of an image \( f \) to be unique, the set of wavelets \( \{W^j_2(x_j, y), W^j_2(x, y_j)\} \) that corresponds to the maxima locations has to constitute a frame in the 2-D signal subspace, i.e. there exist constants \( A, B > 0 \), \( A \leq B < \infty \) such that

\[
A\|f\|^2 \leq \sum_{j \in \mathbb{Z}} \left( \sum_{(x, y)} |<f, \Psi^j_2(x, y)>|^2 \right)^{1/2} \leq B\|f\|^2
\]

(4)
If eqn. 4 is satisfied, then an image can be reconstructed exactly from its WTMM. While it is possible to determine the uniqueness of a 1-D WTMM [3, 4], it is unclear as to how to determine the uniqueness of a 2-D WTMM in practice. A more practical issue, however, is related to the problem of how to reconstruct an image from its WTMM.

The 2-D reconstruction attempts to reconstruct the image by first reconstructing its 2-D dyadic WT from its 2-D WTMM. The image is then obtained by performing a 2-D inverse dyadic WT on the recovered transform.

Let \( K \) be the solution space of the 2-D reconstruction problem, then any element \( \{g^j(x, y), g^j_2(x, y)\}_{j \in \mathbb{Z}} \) belonging to \( K \) must satisfy the following constraints:

1. It must be the 2-D dyadic WT of a function in \( L^2(\mathbb{R}^2) \).
2. At each scale \( 2^j \) and for each modulus maxima location \((x_j, y) \in X_1^j \) and \((x_j, y) \in X_2^j \), we have

\[
g^j_1(x_j, y) = W^j_1(f(x_j, y))
\]

\[
g^j_2(x_j, y) = W^j_2(f(x_j, y))
\]

(5)
3. At each scale \( 2^j \), the 2-D WT modulus maxima obtained from \( g^j_1(x, y) \) and \( g^j_2(x, y) \) are located at the abscissa \((x_j, y) \in X_1^j \) and at the abscissa \((x_j, y) \in X_2^j \), respectively.

The 2-D projection based reconstruction algorithm [2, 8] employs alternating projections onto an affine space \( \Gamma \) specified by the modulus maxima and onto the 2-D dyadic wavelet space \( V \). The space \( \Gamma \) is defined as the affine space of the sequence of functions \( \{g^j(x, y), g^j_2(x, y)\}_{j \in \mathbb{Z}} \) such that for all scales \( 2^j \) and for all modulus maxima locations \((x_j, y) \in X_1^j \) and \((x_j, y) \in X_2^j \), eqn. 5 is satisfied. The 2-D dyadic wavelet space \( V \) is defined as the space of all sequences of functions that satisfy the 2-D reproducing equation. The solution to their reconstruction thus lies on the intersection of \( \Gamma \) and \( \Lambda \), i.e.

\[
\Lambda = \Gamma \cap \Lambda
\]

(6)
It is clear that \( \Lambda \) imposes constraint 1 and \( \Gamma \) imposes constraint 2. The maxima constraint 3 is, however, not imposed by either \( \Gamma \) or \( \Lambda \). There is no guarantee that an element in \( \Lambda \) will only have modulus maxima at the prescribed locations. Any signal having its dyadic WT passing through those wavelet modulus maxima points is by definition in \( \Lambda \) and is considered a mathematically valid solution. One can easily see that the true solution space \( K \) is a subspace of \( \Lambda \).

To "approximately" impose constraint (iii) into the solution, Mallat and Zhong required that: the energy of the reconstructed signal's dyadic WT be as small as possible on average (which will tend to create local modulus maxima only at the prescribed locations) and that the energy of the derivative of the reconstructed signal's dyadic WT be as small as possible on average (which will tend to suppress local modulus maxima at other locations). The above considerations led to the proposal of minimising the Sobolev norm when projecting onto the space \( \Gamma \). The projection operator \( P_\Gamma \) is defined to be the operator that transforms any sequence of functions \( \{g^j_1(x, y), g^j_2(x, y)\}_{j \in \mathbb{Z}} \) into the space of functions \( \{h^j_1(x, y), h^j_2(x, y)\}_{j \in \mathbb{Z}} \) closest to it with respect to the Sobolev norm. Mathematically, this is expressed as

\[
P_\Gamma : \{g^j_1, g^j_2\}_{j \in \mathbb{Z}} \rightarrow \{h^j_1, h^j_2\}_{j \in \mathbb{Z}}
\]

(7)
such that the Sobolev norm of \( \{h^j_1, h^j_2\}_{j \in \mathbb{Z}} = \{h^j_1 - g^j_1, h^j_2 - g^j_2\}_{j \in \mathbb{Z}} \), i.e.

\[
\sum_{j=-\infty}^{\infty} (\|g^j_1\|^2 + \|g^j_2\|^2 + \|h^j_1\|^2 + \|h^j_2\|^2)
\]

(8)
is minimised. Their algorithm is then stated as iterating the composite operator \( P = P_\Gamma P_\Lambda \), i.e. \( \gamma = \lim_{n \rightarrow \infty} P^n \), where \( x \) and \( y \) are the initial estimate and the final solution, respectively. Although \( P_\Gamma \) will tend to create modulus maxima at the prescribed locations while trying to suppress spurious maxima at other locations, the projection onto \( \Gamma \) is no longer a minimum distance projection with respect to the Euclidean norm \( L_2 \) and therefore not necessarily a nonexpansive operator with respect to the \( L_2 \) norm. The iterative algorithm therefore has the potential to diverge in \( L_2 \) norm. In addition, constraint 3 is not necessarily satisfied upon convergence.

Although we found that the reconstruction algorithm of Mallat and Zhong generally produces satisfactory solution in most cases, poor reconstructions can sometime arise. To illustrate, we show in Figs. 1-4 1-D reconstruction of a signal from its WTMM using Mallat and Zhong's

![Fig. 1 Reconstruction using P](image-url)

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original signal
reconstructed signal obtained after 80 iterations where its SNR is at a maximum

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algorithm. For this signal, its WTMM is not unique and exact reconstruction is not possible. The iterative reconstruction is started from the zero sequence. The reconstructed signal shown in Fig. 1 is of a fairly poor quality especially for the square pulse. Fig. 4 shows that Mallat and Zhong’s algorithm exhibits oscillatory behaviour and fails to converge monotonically in $L_2$ norm. By comparing Fig. 2 with Fig. 3, we observe that spurious maxima are present in the WTMM of the final reconstructed signal. The dyadic WT of the reconstructed signal oscillates slightly at locations of abrupt change. This is in agreement with the observation reported by Mallat and Zhong [2], in which they proposed a heuristic way to clip these spurious maxima.

3 The proposed solution

We propose a projection-based reconstruction algorithm that incorporates all the available a priori constraints into the reconstruction such that the solution must only have WT modulus maxima at the locations $X_1$ and $X_2$.

The projection operator onto the linear 2-D dyadic wavelet space $V$ that imposes constraint 1 is given by the reproducing kernel equation

$$P_V = WW^{-1}$$  \hspace{1cm} (9)

Let $C$ be the space that imposes constraints 2 and 3, i.e. $C = \Gamma \cap $ Space for constraint 3. Then the projection $P_C$ can be implemented as follows. First, the values of the estimated 2-D dyadic WT at locations $X_1$ and $X_2$ are replaced by the true values. We could then have two possible situations,

1 A signal which is not monotonically increasing or decreasing between two modulus maxima of opposite sign. A monotonically increasing or decreasing signal that is closest (in $L_2$ sense) to the given signal needs to be constructed.

2 A signal which has more than one local modulus minimum between two adjacent modulus maxima of the same sign. A closest signal that contains only one modulus minimum needs to be constructed. This is done by breaking the original signal into two signals at the point with smallest magnitude and processing the two signals into a non-increasing signal and a non-decreasing signal.

Notice that in situation 2 above, the point at which to break the signal changes from iteration to iteration. This strategy avoids the need to pre-determine a local modulus minimum point, which is not known a priori. Also, it ensures that the new signal is the closest to the original signal in $L_2$ norm.
while enforcing the modulus maxima constraint at each iteration.

Since a separable 2-D dyadic WT is used, the 2-D implementation of \( P_C \) is simplified to two 1-D implementations; one on the rows of \( W_2^\odot g(x, y) \) and the other on the columns of \( W_2^\odot g(x, y) \). In discrete implementation, the projection \( P_C \) can be formulated as a quadratic programming problem as shown next.

### 3.1 An exact implementation of \( P_C \) by QP

Suppose we have a discrete sequence \( \{k_i\}_{i=0, \ldots, N} \) between the positive modulus maximum \( k_0 \) and the negative modulus maximum \( k_N \) that violates the nonincreasing constraint, then the minimum distance projection of \( \{k_i\}_{i=0, \ldots, N} \) onto \( C \) is equivalent to finding a nonincreasing sequence \( \{x_i\}_{i=0, \ldots, N} \) with \( x_0 = k_0 \) and \( x_N = k_N \) that is closest to \( \{k_i\}_{i=0, \ldots, N} \) in the \( L_2 \) norm. The \( L_2 \) distance measure between the sequence \( \{x_i\} \) and \( \{k_i\} \) is given by

\[
d^2 = \sum_{i=1}^{N-1} (x_i - k_i)^2
\]

Minimisation of \( d^2 \) is equivalent to maximisation of

\[-\sum_{i=1}^{N-1} x_i^2 + 2 \sum_{i=1}^{N-1} k_i x_i \]

subject to

- \( x_0 = k_0 \)
- \( x_{i+1} - x_i \leq 0 \) \( i = 1, \ldots, N - 2 \)
- \( x_i \geq 0 \) \( i = 1, \ldots, N - 1 \)

Since the objective function is quadratic and positive semidefinite and the constraints are linear, it is a concave quadratic programming (QP) problem [9]. For a concave QP problem, the optimal solution always exists and satisfies the Kuhn–Tucker (KT) conditions:

Given a QP

\[
\max_{b_i, \cdots, x_j} \left\{ \sum c_i x_i - \sum_{j,k} x_j g_{jk} \right\}
\]

subject to

- \( \sum_j a_{ij} x_j \leq b_i \) \( \forall i \)
- \( x_j \geq 0 \) \( \forall j \)

KT conditions

\[
c_j - 2 \sum_k q_{jk} \hat{x}_k - \sum_i \lambda_i \delta_{ij} + \mu_j = 0 \] \( \forall j \)

\[
\lambda_i \geq 0 \] \( \forall i \)

\[
\mu_j \geq 0 \] \( \forall j \)

\[
\lambda_i \hat{x}_i = 0 \] \( \forall i \)

\[
\mu_j \hat{x}_j = 0 \] \( \forall j \)

where \( \{\hat{x}_i\}, \{\hat{x}_j\} \) are the optimal solution to QP.

The KT conditions transform the QP into a set of linear equations involving equalities

\[
c_j - 2 \sum_k q_{jk} \hat{x}_k - \sum_i \lambda_i \delta_{ij} + \mu_j = 0 \] \( \forall j \)

\[
\sum_j a_{ij} \hat{x}_j + s_i = b_i \] \( \forall i \)

where \( \lambda_i \hat{x}_i = \mu_j \hat{x}_j = 0 \), \( \lambda_i \), \( \mu_j \), \( \hat{x}_i \), \( s_i \), \( b_i \) which can be solved using the simplex method [9]. The composite operator for our algorithm is then given by \( P_1 = P_2 P_3 P_4 \).

Figs 5–7 shows the improvement in the 1-D reconstruction using our algorithm. Convergence is evident and there is no spurious maxima in the WT of the reconstructed signal.

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**Fig. 5** Reconstruction using \( P_1 \)

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original signal

reconstructed signal obtained after 100 iterations

**Fig. 6** Discrete dyadic WT of reconstructed signal in Fig. 5
3.2 An approximate implementation of $P_C$

Unfortunately, the quadratic programming technique is computationally too expensive in a 2-D setting. Instead, we describe a simple approximation that implements the projection $P_C$.

Consider the signal shown in Fig. 8. This signal has one spurious maximum in between two WT modulus maxima $m_1$ and $m_2$. The problem is to find a non-increasing signal between $m_1$ and $m_2$ that is closest to the given signal in $L_2$ norm. Obviously, the new signal will be exactly the same as the given signal except at the interval between $x_2$ and $x_3$. At this interval, the signal should be replaced by a constant value $v$ such that the sum of the two $L_2$ errors $e_1$ and $e_2$ is minimised. We approximate the above problem to finding an optimum combined estimate $v$ from $v_1$ and $v_2$ that has a minimum weighted error. Let $E_1$ and $E_2$ be the $L_2$ error associated with choosing $v = v_1$ and $v = v_2$, respectively. The two errors $E_1$ and $E_2$ are given by

$$E_1 = \int_{x_1}^{x_2} (f(x) - v_1)^2 \, dx$$

$$E_2 = \int_{x_3}^{x_4} (f(x) - v_2)^2 \, dx$$

(15)

If $E_1$ is much smaller than $E_2$, then $v$ should be much closer to $v_1$ than to $v_2$, i.e. the term $(v - v_1)^2$ should be much smaller than $(v - v_2)^2$. This would be achieved if, in finding the least sum of the two terms, we weigh the former term by a larger constant than the latter term. A

---

**Fig. 7** SNRs against number of iterations using $P$ (solid line) and $P_1$ (dotted line)

**Fig. 8** Signal with spurious maximum between modulus maxima $m_1$ and $m_2$

**Fig. 9** Obtaining a nonincreasing signal between two modulus maxima $m_1$ and $m_2$ when spurious maxima are present

**Fig. 10** Original 'Lena' image, 256 x 256 pixels

**Fig. 11** Original 'Peppers' image, 256 x 256 pixels
logical weighting is to weight each term by one-over their associated error

$$E = \frac{(v - v_a)^2}{E_2} + \frac{(v - v_b)^2}{E_1}$$  \hspace{1cm} (16)

In eqn. 16, the two terms are automatically weighted according to their importance. Differentiating eqn. 16 with respect to $v$ and setting the result equal to zero yields

$$v = v_a + \frac{E_2}{E_1 + E_2}(v_b - v_a)$$  \hspace{1cm} (17)

When a signal has more than one spurious maximum, it can be processed as illustrated by the example in Fig. 9 to give a signal with no spurious maxima. A spurious maximum can only be removed if the interval at which $E_1$ or $E_2$ is calculated contains only one minimum or one maximum. This ensures that only one spurious maximum is removed at a time. The magnitude of a signal between the interval $m_1$ and $m_2$ is also required to lie between $v_1$ and $v_2$ before any spurious maximum is smoothed out. This can be done by replacing any value that is larger than $v_1$ by $v_1$ and any value that is smaller than $v_2$ by $v_2$. The following steps are taken when removing multiple spurious maxima:

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**Fig. 12** Four-level dyadic WT and WTMM of 'Lena'

First two columns from the left show, respectively, $W_{11}^1$ and $W_{21}^1$ for scale 1 to 4. Third and fourth columns show the WT modulus maxima obtained from $W_{22}^1$ along the horizontal direction and from $W_{22}^2$ along the vertical direction, respectively. Bottom image is the lowpass approximation at level 4.
1. Remove the left most spurious maximum (marked by an asterisk in Fig. 9) first if possible
2. If the left most spurious maximum cannot be removed, move to the next spurious maximum on the right until a spurious maximum is removed
3. Repeat 1 and 2 until all spurious maxima are removed

In actual implementation, the above procedure is performed on discrete time samples. The $L_2$ norm becomes the $L_1$ norm and the integral becomes summation. The discrete sequence obtained from the above procedure becomes an approximation to the optimum discrete sequence obtained using QP. The approximate sequence was observed to provide a very good approximation to the optimum sequence obtained using QP. Simulation with 1-D signals indicates that reconstruction using the approximate sequence is almost identical to that using the exact implementation of $P_C$ via QP. While our projection algorithm using the exact implementation of $P_C$ is computationally more expensive and hence slower than Mallat and Zhong’s projection algorithm, our projection algorithm using the approximate $P_C$ is comparable in computation speed to their projection algorithm.

**Fig. 13** Reconstruction 'Lena' from four-level subsampled WTMM after 50 iterations
- a Reconstruction using $P$
- b Reconstruction using $P_1$

**Fig. 15** Reconstruction of 'Peppers' from four-level subsampled WTMM after 100 iterations
- a Reconstruction using $P$
- b Reconstruction using $P_1$

**Fig. 14** Reconstruction of 'Lena' from four-level subsampled WTMM after 50 iterations
- SNR against number of iterations
  - $P$
  - $P_1$

**Fig. 16** Reconstruction of 'Peppers' from four-level subsampled WTMM after 100 iterations
- SNR against number of iterations
  - $P$
  - $P_1$
4 Simulation results

To compare our proposed algorithm with Mallat and Zhong's algorithm, we present some reconstruction results for two images shown in Figs 10 and 11. Fig 12 shows the four-level dyadic WT and WTMM of 'Lena'. Since the lowpass approximation has most of its energy concentrated in a bandwidth that is approximately 16 times smaller than the bandwidth of the original signal in the horizontal and vertical directions, it can be subsampled by a factor of $2^4$ in both directions, i.e., the $256 \times 256$ image is subsampled to $16 \times 16$. To reconstruct the image from its subsampled WTMM, the subsampled lowpass approximation needs to be upsampled back to the original dimension first. In these examples, simple linear interpolation between samples is used for upsampling.

The reconstructed images of 'Lena' obtained after 50 iterations and the reconstructed images of 'Peppers' obtained after 100 iterations using $P$ and $P_1$ are shown in Figs 13–16. For the image 'Lena', both $P$ and $P_1$ gave results that are visually identical to the original image after 50 iterations. The divergence of Mallat and Zhong's algorithm in $L_2$ norm is evidence in the reconstruction for 'Lena'. For the image 'Peppers', the reconstruction using $P_1$ has better contrast than the reconstruction using $P$. The reconstruction using $P$ has better convergence initially and this is because the operator $P_1$ in $P$ projects to a sequence that is closer to the true sequence initially.

5 The accelerated algorithm

It is possible to accelerate the convergence of the projection algorithm $P_1$ by exploiting the similarity of the difference image between each iteration. Let $f_i$ be the reconstructed image at the $i$th iteration, the difference image $d_i$ is defined by

$$d_i = f_{i+1} - f_i$$

Fig. 17 shows the difference images $d_1, d_2, d_3, d_4$ for the reconstruction from the subsampled WTMM of 'Lena' using $P_1$. The difference images $d_1$ and $d_2$ are very similar for both 'Lena' and 'Peppers'. A quantitative measure of the similarity of the difference images can be obtained by using the following similarity metrics $S$

$$S(d_i, d_{i-1}) = \frac{<d_i, d_{i-1}>}{\|d_i\|\|d_{i-1}\|}$$

The value of $S$ ranges between 0 and 1. When $S = 1$, $d_i$ and $d_{i-1}$ are identical to each other. Table 1 lists the value of $S$ between the difference images for the reconstruction from the subsampled WTMM of 'Lena' and 'Peppers' using $P_1$. The similarity between difference images $d_i$ and $d_{i-1}$ is greater than 0.9 for $i$ greater than 2. In fact, $S$ approaches 1 as $i$ increases. When $S \approx 1$, $f_i \approx f_{i-1} + (f_{i-1} - f_{i-2})$, i.e., $f_i$ can be obtained from the weighted sum of $f_{i-1}$ and $f_{i-2}$ instead of from the projection $P_1$ on $f_{i-1}$. Therefore, the reconstruction can be accelerated by the incorporation of a momentum term $\alpha$, i.e.

$$f_i = f_i' + \alpha(f_i' - f_{i-1})$$

where $f_i$ and $f_{i-1}$ are the current reconstruction and the previous reconstruction, respectively, and $f_i' \approx P_1f_{i-1}$. The momentum term $\alpha$ lies between 0 and 1 and it is usually chosen to be roughly equal to $S$. This acceleration method tries to build inertia into the estimation, and thus the next estimate is predicted using the information from the
previous estimate. Significant improvement in convergence rate based on the momentum idea has also been demonstrated [10] for blind deconvolution and in neural network weight update [11].

For the reconstruction of ‘Lena’ and ‘Peppers’ from their subsampled WTMM using $P_a$, $\alpha > 0$ gives improvement on the unaccelerated reconstruction and $\alpha = 1$ gives the best result. This is in accordance with the similarity values presented in Table 1. In general, we found that a value of $\alpha = 1$ will give near optimum reconstruction performance for most images. Fig. 18 shows the SNRs of the reconstructions from the subsampled WTMM of ‘Lena’ and ‘Peppers’, respectively, using $P_1$ with $\alpha = 1$. The acceleration effect can be readily observed.

The reconstruction algorithm using $P_a$ is relatively stable with respect to perturbation in the WTMM. In one experiment, we linearly quantised the modulus maxima and the lowpass approximation in the WTMM of ‘Lena’ and ‘Peppers’ to 4 bits/pixel. Due to quantisation error, the quantised subsampled WTMM of ‘Lena’ and ‘Peppers’ have a SNR of 18.5 dB and 18.7 dB, respectively. The reconstructed ‘Lena’ and ‘Peppers’ obtained after 20 iterations have a SNR of 33.41 dB and a SNR of 31.16 dB, respectively. The reconstructions are very accurate even though the representations have a fairly low SNR.

6 Conclusions

We have proposed a new projection-based algorithm to reconstruct an image from its WTMM. In contrast to Mallat and Zhong’s algorithm, our algorithm incorporates all the a priori constraints in the WTMM into the reconstruction. Since one of the projection operators, $P_a$, which projects a sequence into another sequence that satisfies the modulus maxima constraint, is computationally expensive to implement exactly using the method of QP, we provide an approximate implementation which gives an almost identical result. We also proposed a way to accelerate the reconstruction algorithm based on the idea of momentum. By exploiting the similarity of the difference images between two consecutive iterations, we are able to use information from the previous estimates to predict the new estimate, thus improving the convergence rate. Simulation results indicated that good solutions can be obtained from our algorithm.

7 References