# Fundamentals of discrete-time signal processing

Objective of this chapter:

To focus attention on some important issues of discrete-time signal processing that are of fundamental importance to signal processing.



**Discrete-time signals:**

- Signals: physical quantities that change as a function of time, space, or some other dependent variable.
- Analysis of signals require mathematical signal models that allow one to choose the appropriate mathematical approach for analysis.
- Signal characteristics and the classification of signals based upon either such characteristics or the associated mathematical models are the subject of this Section.

Continuous-time, discrete-time and digital signals

- real-valued / complex-valued signal : depends on the value of the dependent variable
- continuous / discrete : every signal variable may take on values from either a continuous set of values or a discrete set of values:

| | Dependent variable | Independent variable |
|---|---|---|
| Continuous-time signal | Continuous | Continuous |
| Digital signal | Discrete | Discrete |
| Discrete signal | Don't care | Discrete |

- Discrete signal is our concern in this class.



Mathematical description of signals

- The mathematical analysis of a signal requires the availability of a mathematical description for the signal itself.
- The description is usually referred to as a *signal model*.
- In the book, the term signal is used to refer to either the signal itself or its model.

*Deterministic signals*

- Any signal that can be described by an explicit mathematical relationship is called deterministic.
- Some basic signals:
  - Unit impulse sequence: $\delta(n)$=1 if n=0; $\delta(n)$=0 else
  - Unit step sequence: u(n)=0 if n<0; u(n)=1 else
  - Exponential sequence of the form: x(n)=$a^n$
- Signal classification:
  Deterministic signals can be classified as (1) energy or power, (2) periodic or periodic, (3) of finite or infinite duration, (4) causal or non-causal, and (5) even or odd signals.



- Energy signal vs. Power signal:

| | Energy | Power |
|---|---|---|
| Energy signal | Finite, $0 < E_x < \infty$ (def) | =0 |
| Power signal | $= \infty$ | Finite, $0 < P_x < \infty$ (def) |

  - Energy of a signal : $E_x = \sum_{-\infty}^{\infty} x(n)^2 \geq 0$
  - Power of a signal : $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} x(n)^2 \geq 0$

- A discrete-time signal x(n) is *periodic* with fundamental period N if x(n+N)=x(n) for all N. Otherwise it is called *aperiodic*.
- A periodic signal is a power signal.
- A signal x(n) has finite duration if x(n)=0 for n<N1 and n>N2, where N1 and N2 are finite integer numbers and N1≤N2. If N1=-∞ and/or N2=∞, it has infinite duration.
- A signal x(n) is said to be *causal* if x(n)=0 for n<0. Otherwise it is *noncausal*.
- A real-valued signal x(n) is *even* if x(-n)=x(n) and *odd* if x(-n)=-x(n).

## *Random signals*

- Signals that can't be described to any reasonable accuracy by explicit mathematical relationships are called random signals.
- Though random signals are evolving in time in an unpredictable manner, their average properties can often be assumed to be deterministic.
- Random signals are thus mathematically described by stochastic processes and can be analyzed by using statistical methods instead of explicit equations.
- The theory of probability, random variables, and stochastic processes provides the mathematical framework for the theoretical study of random signals.

Real-world signals:

- In practical terms, the decision as to whether physical data are deterministic or random is usually based upon the ability to reproduce the data by controlled experiments.



## Transform-domain representation of deterministic signals

- Deterministic signals are assumed to be explicity known for all time, so the simplest description of any signal is an amplitude-versus-time plot.
- Frequency analysis is the process of decomposing a signal into frequency components.
- Two characteristics that specifies the analysis tools:
  (1) The nature of time: continuous-time or discrete-time signals.
  (2) The existence of harmony: periodic or aperiodic signals

| | Periodic | Aperiodic |
|---|---|---|
| Continuous-time | Fourier series $X(k)=\frac{1}{T}\int_0^T x(t)e^{-j2\pi kt/T}dt$ $x(t)=\sum_{k=-\infty}^{\infty}X(k)e^{j2\pi kt/T}$ | Fourier Transform $X(f)=\int_{t=-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$ $x(t)=\int_{f=-\infty}^{\infty}X(f)e^{j2\pi ft}df$ |
| Discrete-time | Fourier series (c.w. DFT) $X_k=\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j(2\pi/N)kn}$ $x(n)=\sum_{k=0}^{N-1}X_k e^{j(2\pi/N)kn}$ | Fourier transform $X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}$ $x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$ |



- Spectral classification:

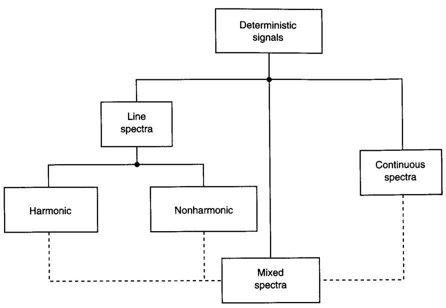



**FIGURE 2.1**
Spectral classification of deterministic (finite power or energy) signals.

Sampling of continuous-time signals

- In most practical applications, discrete-time signals are obtained by sampling continuous-time signals periodically in time.
- Sampling frequency/rate: the number of samples taken per unit of time ($=F_s$)
- Sampling period: $1/F_s$



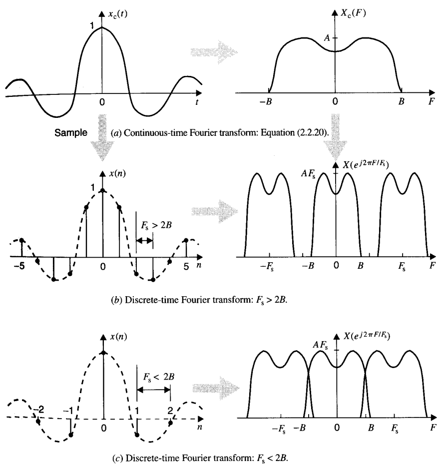



**FIGURE 2.2** Sampling operation.

- Sampling theorem:
  In order to avoid *aliasing*, the sampling rate must be at least equal to twice the bandwidth of a band-limited, real-valued, continuous-time signal.
- The minimum sampling rate of $F_s = 2B$ is called *Nyquist rate*.

## The discrete Fourier transfrom

- The discrete Fourier transform (DFT) of a sequence x(n) and the corresponding inverse discrete Fourier transform (IDFT) are, respectively, given by:

$$X_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j(2\pi/N)kn}$$

(c.w. the analysis tool for discrete-time periodic deterministic signal.)

## The z-transform

- The z-transform of a sequence x(n) and the corresponding inverse z-transform are, respectively, given by:

$$X(z) \equiv \mathcal{Z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (2.2.29)$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- The set of values of z for which (2.2.29) converges is called the *region of convergence* (ROC) of X(z).



- The contour of integration in the inverse z-transform can be any counterclockwise closed path that encloses the origin and is inside the ROC.

- Connection between z-transform & DFT:

$$X(z)\big|_{x=e^{j\omega}} = X(e^{j\omega})$$

- Properties of z-transform

**TABLE 2.1**
**Properties of z-Transform.**

| Property | Time domain | z-Domain | ROC |
|---|---|---|---|
| Notation | $x(n)$ | $X(z)$ | ROC : $R_l < \|z\| < R_u$ |
| | $x_1(n)$ | $X_1(z)$ | $ROC_1 : R_{1l} < \|z\| < R_{1u}$ |
| | $x_2(n)$ | $X_2(z)$ | $ROC_2 : R_{2l} < \|z\| < R_{2u}$ |
| Linearity | $a_1x_1(n) + a_2x_2(n)$ | $a_1X_1(z) + a_2X_2(z)$ | $ROC_1 \cap ROC_2$ |
| Time shifting | $x(n-k)$ | $z^{-k}X(z)$ | $R_l < \|z\| < R_u$, except $z = 0$ if $k > 0$ |
| Scaling in the z-domain | $a^n x(n)$ | $X(a^{-1}z)$ | $\|a\|R_l < \|z\| < \|a\|R_u$ |
| Time reversal | $x(-n)$ | $X(z^{-1})$ | $\frac{1}{R_l} < \|z\| < \frac{1}{R_u}$ |
| Conjugation | $x^*(n)$ | $X^*(z^*)$ | ROC |
| Differentiation | $nx(n)$ | $-z\frac{dX(z)}{dz}$ | ROC |
| Convolution | $x_1(n) * x_2(n)$ | $X_1(z)X_2(z)$ | $ROC_1 \cap ROC_2$ |
| Multiplication | $x_1(n)x_2(n)$ | $\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$ | $R_{1l}R_{2l} < \|z\| < R_{1u}R_{2u}$ |
| Parseval's relation | $\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$ | | |

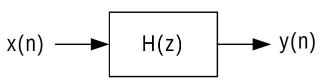


## **Discrete-time systems**

- In this section, we review the basics of linear, time-invariant systems.

- A *system* is defined to be any physical device or algorithm that transform a signal, called the *input* or *excitation*, into another signal, called the *output* or *response*.

- The mathematical relationships between the input and output signals of a system is referred to as a *system model*.




Block diagram representation of a discrete-time system

## Analysis of linear, time-invariant (LTI) systems

- The systems we deal with are linear and time-invariant and are always assumed to be initially at rest.



### *Time-domain analysis*

- The output of a linear, time invariant system can always be expressed as the convolution summation between the input sequence x(n) and the impulse response sequence h(n) of the system.

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- In matrix form, we've

$$\begin{bmatrix} y(0) \\ \vdots \\ y(M-1) \\ \vdots \\ y(N-1) \\ \vdots \\ y(L-1) \end{bmatrix} = \begin{bmatrix} x(0) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ x(M-1) & \cdots & \cdots & x(0) \\ \vdots & \ddots & \ddots & \vdots \\ x(N-1) & \cdots & \cdots & x(N-M) \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x(N-1) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}$$

or

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(L-1) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(M-2) \\ 0 & 0 & \cdots & h(M-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Toeplitz matrix: all the elements along any diagonal are equal.

- A system is called *causal* if the present value of the output signal depends only on the present and/or past values of the input signal. (i.e. h(n)=0 for n<0)
- A system is called *stable* if every bounded input produces a bounded output.
(i.e. $x(n) < \infty \Rightarrow y(n) < \infty$ for all n)
- An LTI system is stable iff $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.
- A system has an impulse response with finite duration is called a *finite impulse response* (FIR) system. Otherwise, it's called an *infinite impulse response* (IIR) system.



*Transform domain analysis*

- $y(n) = h(n) * x(n) \quad \Leftrightarrow \quad$ Y(z)=H(z)X(z)
- A causal discrete-time system can also be described with a linear difference equation.

$$y(n) = -\sum_{k=1}^{P} a_k y(n-k) + \sum_{k=0}^{Q} d_k x(n-k)$$

- If system parameters $\{a_k, d_k\}$ depend on time, the system is *time-varying*. Otherwise, it's *time-invariant*.
- If system parameters $\{a_k, d_k\}$depend on either the input or output signals, the system is *nonlinear*. Otherwise, it's *linear*.
- With z-transform, we've

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{Q} d_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}} \equiv \frac{D(z)}{A(z)}$$

It can be rewritten as

$$H(z) = \frac{D(z)}{A(z)} = G \frac{\prod_{k=1}^{Q} (1 - z_k z^{-1})}{\prod_{k=1}^{P} (1 - p_k z^{-1})}$$



- The roots of D(z) and A(z) are, respectively, referred to as *zeros* and *poles* of the system.

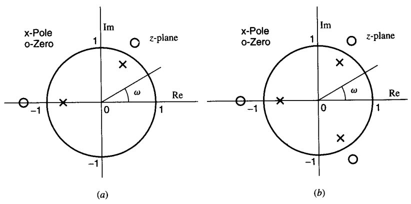



**FIGURE 2.10**
Typical pole-zero patterns of a PZ, all-pass system: (*a*) complex-valued coefficients and (*b*) real-valued coefficients.

- The system is *stable* if its poles are all inside the unit cycle.

All-zero (AZ) system: (P=0)

- $y(n) = \sum_{k=0}^{Q} d_k x(n-k)$
- $H(z) = \sum_{k=0}^{Q} d_k z^{-k}$
- $h(n) = \begin{cases} d_n & 0 \le n \le Q \\ 0 & elsewhere \end{cases}$

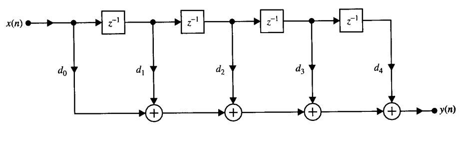



**FIGURE 2.6** FIR filter realization (direct form).



All-pole (AP) system: (Q=0)

- $y(n) = -\sum_{k=1}^{P} a_k y(n-k) + x(n)$
- $H(z) = \frac{1}{1 + \sum_{k=1}^{P} a_k z^{-k}} = \sum_{k=1}^{P} \frac{A_k}{1 - p_k z^{-k}}$
- $h(n) = \sum_{k=1}^{P} A_k \, (p_k)^n u(n)$
- Any nontrivial pole in a system implies an infinite duration impulse response.

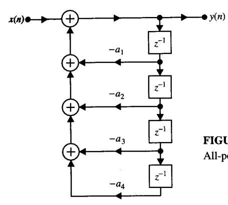



**FIGURE 2.7**
All-pole system realization (direct form).

## Pole-zero (PZ) system: (P>0 & Q>0)

- $y(n) = -\sum_{k=1}^{P} a_k y(n-k) + \sum_{k=0}^{Q} d_k x(n-k)$
- $H(z) = \dfrac{\sum_{k=0}^{Q} d_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}}$

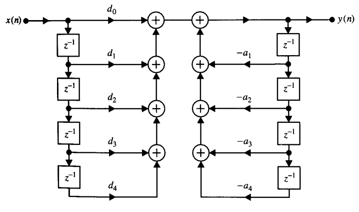



**FIGURE 2.8** Pole-zero system realization (direct form).

- Summary:

| Model | AZ | AP | PZ |
|---|---|---|---|
| Condition | P=0 | Q=0 | P,Q>0 |
| Property | FIR | IIR | IIR |



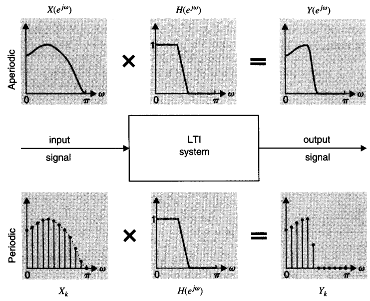



**FIGURE 2.9** LTI system operation in the frequency domain.

## Correlation analysis and spectral density

- The investigation of system responses to specific input signals requires either
  (1) the explicit computation of the output signal or
  (2) measurements to relate characteristic properties of the output signal to corresponding characteristics of the system and the input signal.

- Correlation between two signals provides a quantitative measure of similarity between two signals.

- Connection between correlation & convolution:

| | |
|---|---|
| Convolution | $x(n) * y(n)$ |
| Correlation | $x(n) * y(-n)$ |



- Correlation between two discrete-time signals x(n) and y(n): (*crosscorrelation*)

$$r_{xy}(l) = \begin{cases} \sum_{n=-\infty}^{\infty} x(n) y^*(n-l) & : energy\ signal \\ \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n) y^*(n-l) & : power\ signal \end{cases}$$

- Correlation between signal x(n) and its shifted version (i.e.x(n)=y(n)) (*autocorrelation*)

$$r_x(l) = \begin{cases} \sum_{n=-\infty}^{\infty} x(n) x^*(n-l) & : energy\ signal \\ \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n) x^*(n-l) & : power\ signal \end{cases}$$

- The autocorrelation sequence $r_x(l)$ and the energy spectrum of a signal x(n) form a Fourier transform pair.

$$r_x(l) \xleftrightarrow{F} R_x(e^{j\omega})$$

- The energy of a signal can be determined:
  (1) $E_x = r_x(0)$
  (2) $E_x = \int R_x(e^{j\omega}) d\omega$

- Summary of the usage of correlation:
  (1) Tell the similarity between 2 signals
  (2) Estimate the spectral density of a signal



- Connection between $r_x(l)$ and $r_y(l)$:

$$\begin{aligned} r_y(l) &= y(l) * y^*(-l) = h(l) * x(l) * h^*(-l) * x^*(-l) \\ &= h(l) * h^*(-l) * x(l) * x^*(-l) \\ &= r_h(l) * r_x(l) \end{aligned}$$

where $r_h(l) = \sum_{n=-\infty}^{\infty} h(n) h^*(n-l) = h(l) * h(-l)$

- In z-transform domain, we have

$$\begin{aligned} R_y(z) &= Y(z) Y^*(\tfrac{1}{z}) = H(z) X(z) H^*(\tfrac{1}{z}) X^*(\tfrac{1}{z}) \\ &= H(z) H^*(\tfrac{1}{z}) X(z) X^*(\tfrac{1}{z}) = R_h(z) R_x(z) \end{aligned}$$

where $R_h(z) = H(z) H^*(\tfrac{1}{z})$

(c.w. $R_x(z) = X(z) X^*(\tfrac{1}{z})$)

- Some other direct results: $R_{xy}(z) = H(z) R_x(z)$

## Minimum phase and system invertibility:

- This section introduces the concept of minimum phase and shows how it is related to the invertibility of LTI systems.

System invertibility and minimum-phase systems

- A system H(z) with input x(n) and output y(n) is called invertible if one can uniquely determine its input signal from the output signal. (i.e. A(z)=1/H(z) exists s.t. A(z)H(z)=H(z)A(z)=1)
- A discrete-time, linear, time-invariant system with impulse response h(n) is called minimum-phase if both the system and its inverse system are causal and stable. (i.e. All poles and zeros of H(z) are inside the unit circle.)
- A maximum-phase system is one in which both the system and its inverse are noncausal and stable. (i.e. All poles and zeros are outside the unit circle.)
- If H(z) is minimum-phase, then 1/H(z) is also minimum-phase.
- If H(z) is minimum-phase, then H(1/z) is maximum-phase.
- A system that is neither minimum-phase nor maximum-phase is called a mixed-phase system.



**EXAMPLE** Consider the following all-zero minimum-phase system:

$$H_{\min}(z) = (1-0.8e^{j0.6\pi}z^{-1})(1-0.8e^{-j0.6\pi}z^{-1}) \times (1-0.8e^{j0.9\pi}z^{-1})(1-0.8e^{-j0.9\pi}z^{-1})$$

Determine the maximum- and mixed-phase systems with the same magnitude response.

***Solution.*** To obtain a maximum-phase system with the same magnitude response, we reflect the zeros of $H_{\min}(z)$ from inside the unit circle to their conjugate reciprocal locations that are outside the unit circle by using the transformation $z_0 \to 1/z_0^*$. This leads to the following transformation for each first-order factor:

$$(1-re^{j\theta}z^{-1}) \to r(1-\frac{1}{r}e^{j\theta}z^{-1}) \qquad (2.4.25)$$

The scaling factor $r$ in the right-hand side is included to guarantee that the transformation does not scale the magnitude response. The resulting maximum-phase system is

$$H_{\max}(z) = (0.8)^4(1-1.25e^{j0.6\pi}z^{-1})(1-1.25e^{-j0.6\pi}z^{-1}) \times (1-1.25e^{j0.9\pi}z^{-1})(1-1.25e^{-j0.9\pi}z^{-1})$$

If we reflect only the zero at $0.8e^{\pm j0.6\pi}$, we obtain the mixed-phase system

$$H_1(z) = (0.8)^2(1-1.25e^{j0.6\pi}z^{-1})(1-1.25e^{-j0.6\pi}z^{-1}) \times (1-0.8e^{j0.9\pi}z^{-1})(1-0.8e^{-j0.9\pi}z^{-1})$$

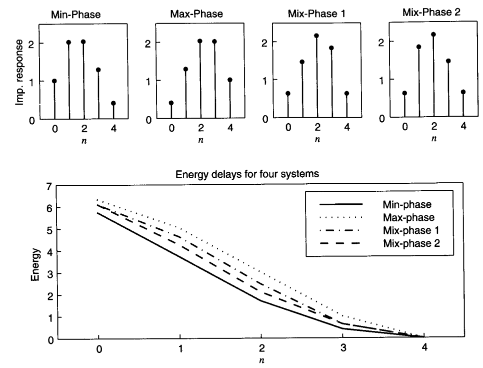



**FIGURE 2.14**
Impulse response plots of the four systems in the top row and the energy delay plots in the bottom row in Example.



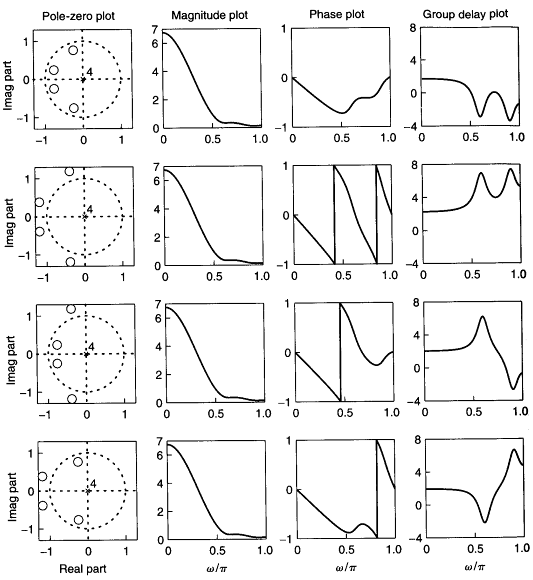



**FIGURE 2.13**
Pole-zero and frequency response plots for minimum-phase (row 1), maximum-phase (row 2), mixed-phase 1 (row 3), and mixed-phase 2 (row 4) systems in Example 2.4.3. Note that the abscissa in Phase plots are labeled in units of $\pi$ radians while those in Group delay plots are labeled in sampling intervals.



All-pass systems

- A linear, time-invariant system is all-pass if $|H(e^{j\omega})| = 1$ for $-\pi < \omega \le \pi$.
- Typical transfer function:

$$H(z) = \frac{a_p^* + a_{p-1}^* z^{-1} + \cdots + z^{-P}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} = \frac{z^{-p}A^*(1/z)}{A(z)}$$

or $$H(z) = \prod_{k=1}^{p} \frac{(p_k^* - z^{-1})}{(1 - p_k z^{-1})}$$

- The poles and zeros of an all-pass system are conjugate reciprocals of one another.

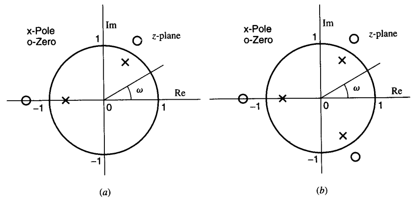



**FIGURE 2.10**
Typical pole-zero patterns of a PZ, all-pass system: (*a*) complex-valued coefficients and (*b*) real-valued coefficients.

Minimum-phase and all-pass decomposition

- Any causal PZ system that has no poles or zeros on the unit circle can be expressed as
  $H(z) = H_{\min}(z) H_{ap}(z)$,
  where $H_{\min}(z)$ and $H_{ap}(z)$ are, respectively, a minimum-phase system and an all-pass system.

Spectral factorization

- The process of obtaining the minimum-phase system that produces the signal y(n) with autocorrelation $r_y(l)$ is called *spectral factorization*.

- Example 2.4.5