



# Z-Transform

- Using Fourier transform for LTI system analysis is tedious since it requires the input and the system coefficients be converted to **complex numbers**
- Most inputs and system coefficients are real
- Need a tool works on real number system  
⇒ **Z-transform**



## Definition

- If a signal  $x[n]$  such that  $\{n = 0, 1, \dots, N-1\}$ , **Z-transform** of  $x[n]$  is defined as follows:

$$X(z) = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{N-1} x[n]z^{-n}$$

- For example, if  $x[n] = \{1 \ 3 \ 2 \ 2 \ 1\}$ ,

$$X(z) = Z\{x[n]\} = 1 + 3z^{-1} + 2z^{-2} + 2z^{-3} + z^{-4}$$

We just map  
the input to a  
polynomial!



## Inverse Z-transform

- Hence if the Z-transform of a signal is given, we can easily convert it back to the time domain sequence

- For example, if

$$X(z) = Z\{x[n]\} = 1 + 3z^{-1} + 2z^{-2} + 2z^{-3} + z^{-4}$$

$$Z^{-1}\{X(z)\} \text{ is equal to } x[n] = \{1 \ 3 \ 2 \ 2 \ 1\}$$

- This is the so-called **inverse Z-transform**



# Properties

## Linearity

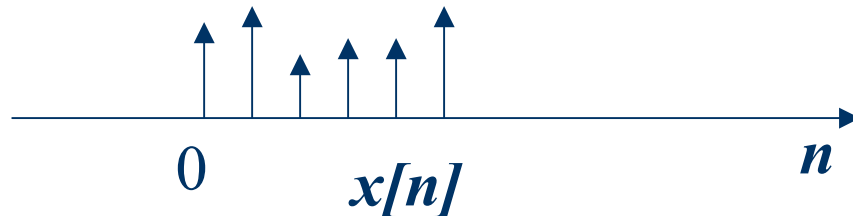
If  $x[n] = x_1[n] + x_2[n]$  for  $n = \{0, 1, \dots, N-1\}$   
and  $a$  and  $b$  are two constants

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} (ax_1[n] + bx_2[n])z^{-n} \\ &= a \sum_{n=0}^{N-1} x_1[n]z^{-n} + b \sum_{n=0}^{N-1} x_2[n]z^{-n} \\ &= aX_1(z) + bX_2(z) \end{aligned}$$

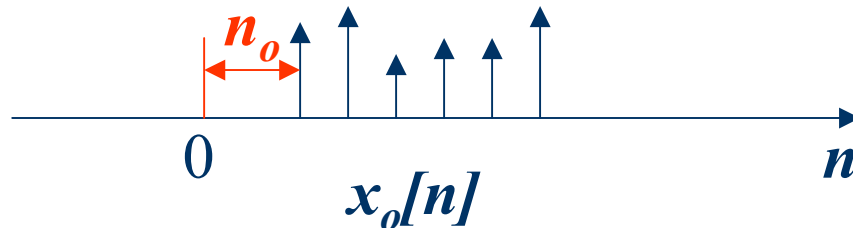


## Time Delay

- Given a signal  $x[n]$  such that  $\{n = 0, 1, \dots, N-1\}$  and  $x[n] = 0$  if  $n < 0$



- Let  $x_o[n]$  be equal to  $x[n]$  but delay by  $n_o$  unit such that  $x_o[n] = x[n - n_o]$





7. Convolution, FIR Filters and Z-Transform

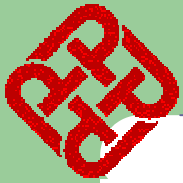
$$X_o(z) = Z\{x_o[n]\} = \sum_{n=0}^{\infty} x[n - n_o] z^{-n}$$

- Let  $n' = n - n_o$ . Hence  $n = n' + n_o$   
 $n = 0 \text{ to } \infty \Rightarrow n' = -n_o \text{ to } \infty$

$$X_o(z) = \sum_{n'=-n_o}^{\infty} x[n'] z^{-(n'+n_o)} = z^{-n_o} \sum_{n'=-n_o}^{\infty} x[n'] z^{-n'}$$

- Since  $x[n] = 0$  if  $n < 0$

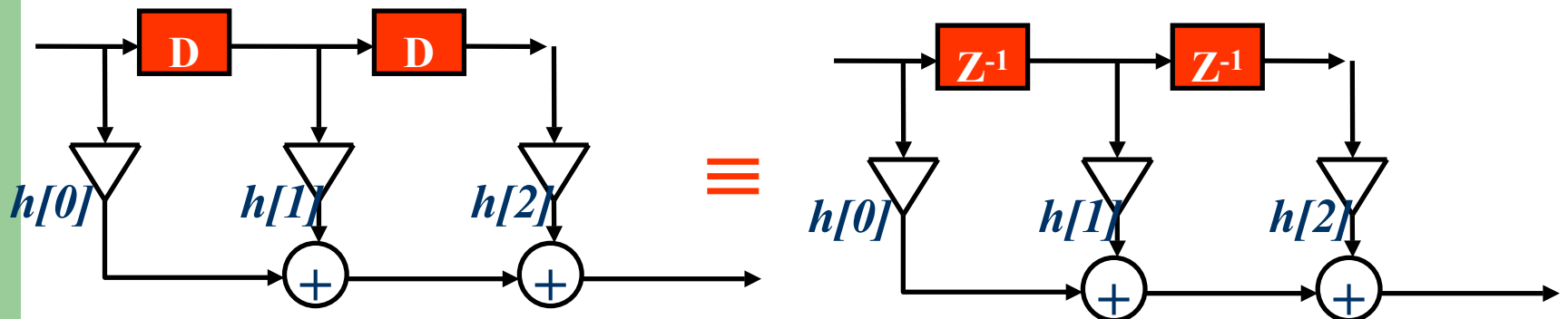
$$X_o(z) = z^{-n_o} \sum_{n'=0}^{\infty} x[n'] z^{-n'} = z^{-n_o} X(z)$$



Signal Processing Fundamentals – Part I  
Spectrum Analysis and Filtering

7. Convolution, FIR Filters and Z-Transform

- If  $x[n]$  is delayed by 1 time unit,  $X_o(z) = z^{-1}X(z)$





# Linear Convolution in Z-domain

- Z-transform is so useful because it **gives a very simple representation of convolution**
- Recall that if we have two finite length sequences  $x[n]$  and  $h[n]$  such that  $n = \{0, 1, \dots, N-1\}$ , their convolution is given as follows:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

- The length of  $y[n]$  is  $2N-1$



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$$\begin{aligned} Z\{y[n]\} &= Z\{x[n] * h[n]\} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{N-1} h[m] x[n-m] z^{-n} \\ &= \sum_{m=0}^{N-1} h[m] \sum_{n=0}^{\infty} x[n-m] z^{-n} \\ &= \sum_{m=0}^{N-1} h[m] z^{-m} X(z) = H(z) \cdot X(z) \end{aligned}$$

Time	Z-domain
Convolution	Multiplication



- Z-transform converts a **linear convolution** to a **polynomial multiplication**
- It does not reduce the number of operations
- But it gives an **algebraic explanation** to linear convolution
- By using different tools in algebra, we can analyze our system in a more efficient way



- **Example**

If  $x[n] = \{1 \ 2 \ 3\}$  and  $h[n] = \{1 \ 1 \ 1\}$ ,  $y[n] = x[n] * h[n]$

$$X(z) = 1 + 2z^{-1} + 3z^{-2}; \quad H(z) = 1 + z^{-1} + z^{-2}$$

$$X(z).H(z) = (1 + 2z^{-1} + 3z^{-2}).(1 + z^{-1} + z^{-2})$$

$$= 1 + z^{-1} + z^{-2} + 2z^{-1} + 2z^{-2} + 2z^{-3}$$

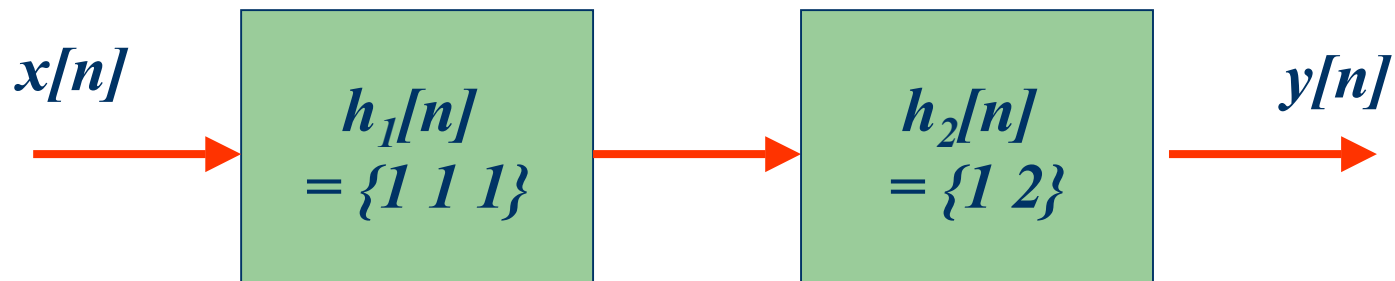
$$+ 3z^{-2} + 3z^{-3} + 3z^{-4}$$

$$= 1 + 3z^{-1} + 6z^{-2} + 5z^{-3} + 3z^{-4}$$

$$y[n] = Z^{-1}\{X(z).H(z)\} = \{1 \ 3 \ 6 \ 5 \ 3\}$$

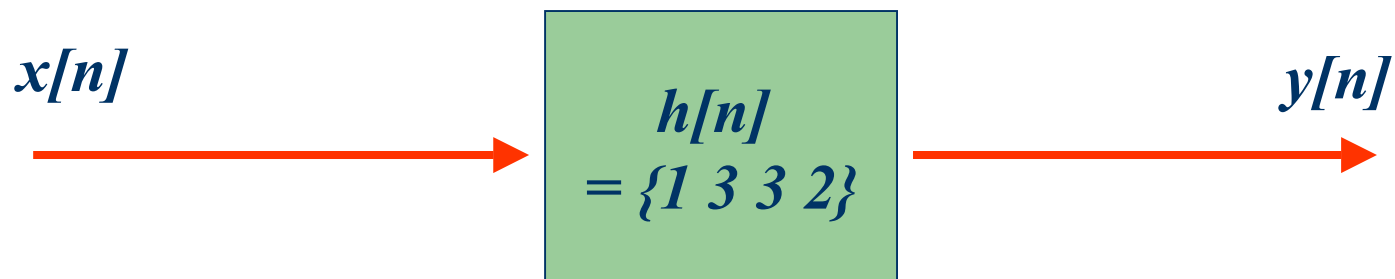


# Cascading LTI Systems



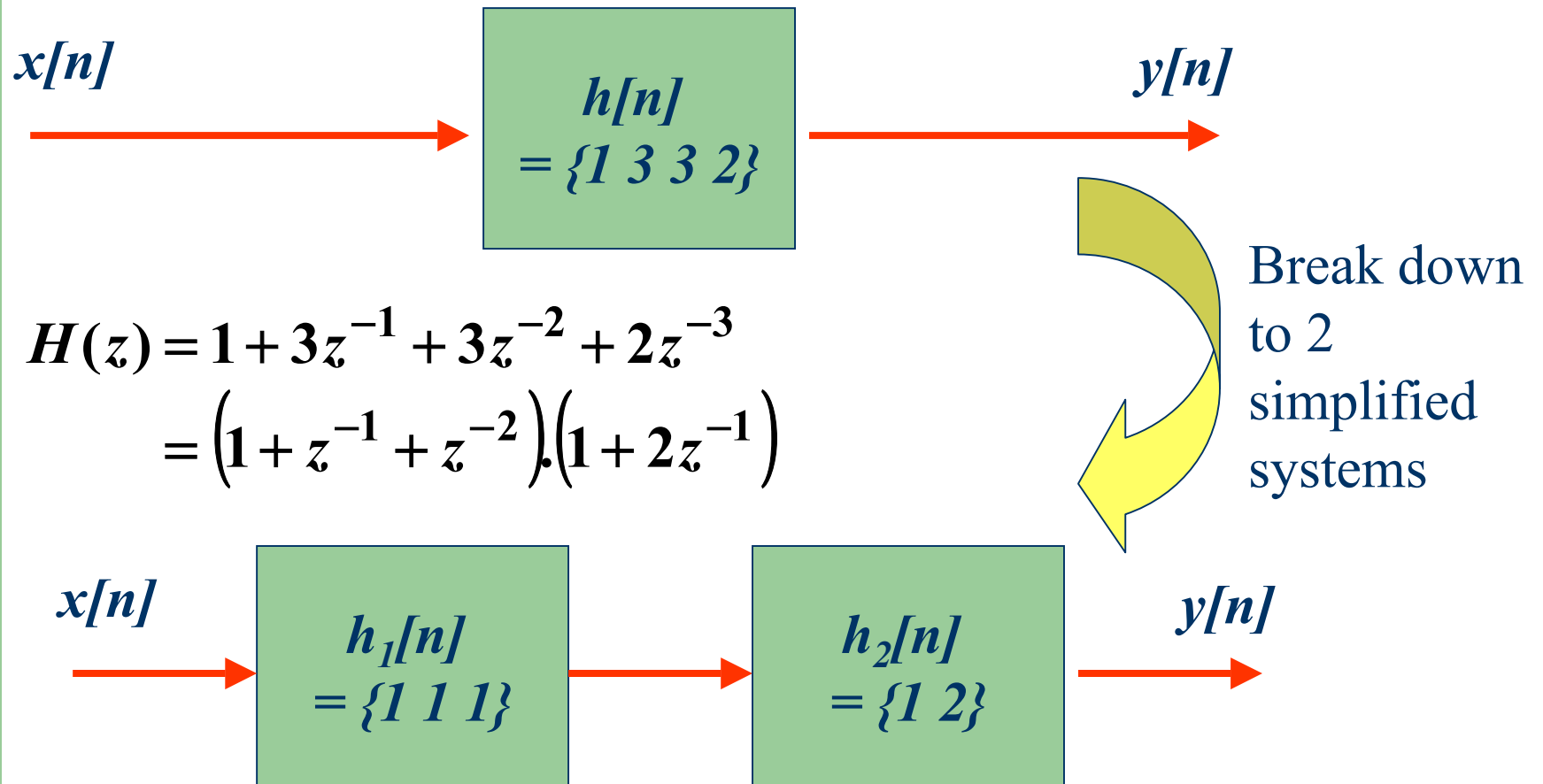
$$Y(z) = X(z) \cdot H_1(z) \cdot H_2(z) \\ = X(z) \cdot H(z)$$

$$H(z) = (1 + z^{-1} + z^{-2}) \cdot (1 + 2z^{-1}) \\ = 1 + 3z^{-1} + 3z^{-2} + 2z^{-3}$$





# Factorizing an LTI System





## Relationship with Fourier Domain

- Although Z-transform gives efficient representation to signal, in many cases we need to know the frequency (or Fourier) response of the system
- We need to find out the **relationship between Fourier-domain and Z-domain**
- Hence when we work on Z-domain, we know what will happen to the frequency response of the system



- If an LTI system has an impulse response  $h[n]$ ,  $n = \{0, 1, \dots, N-1\}$ ,

**Fourier Domain**

$$H_p(\hat{\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\hat{\omega}n}$$

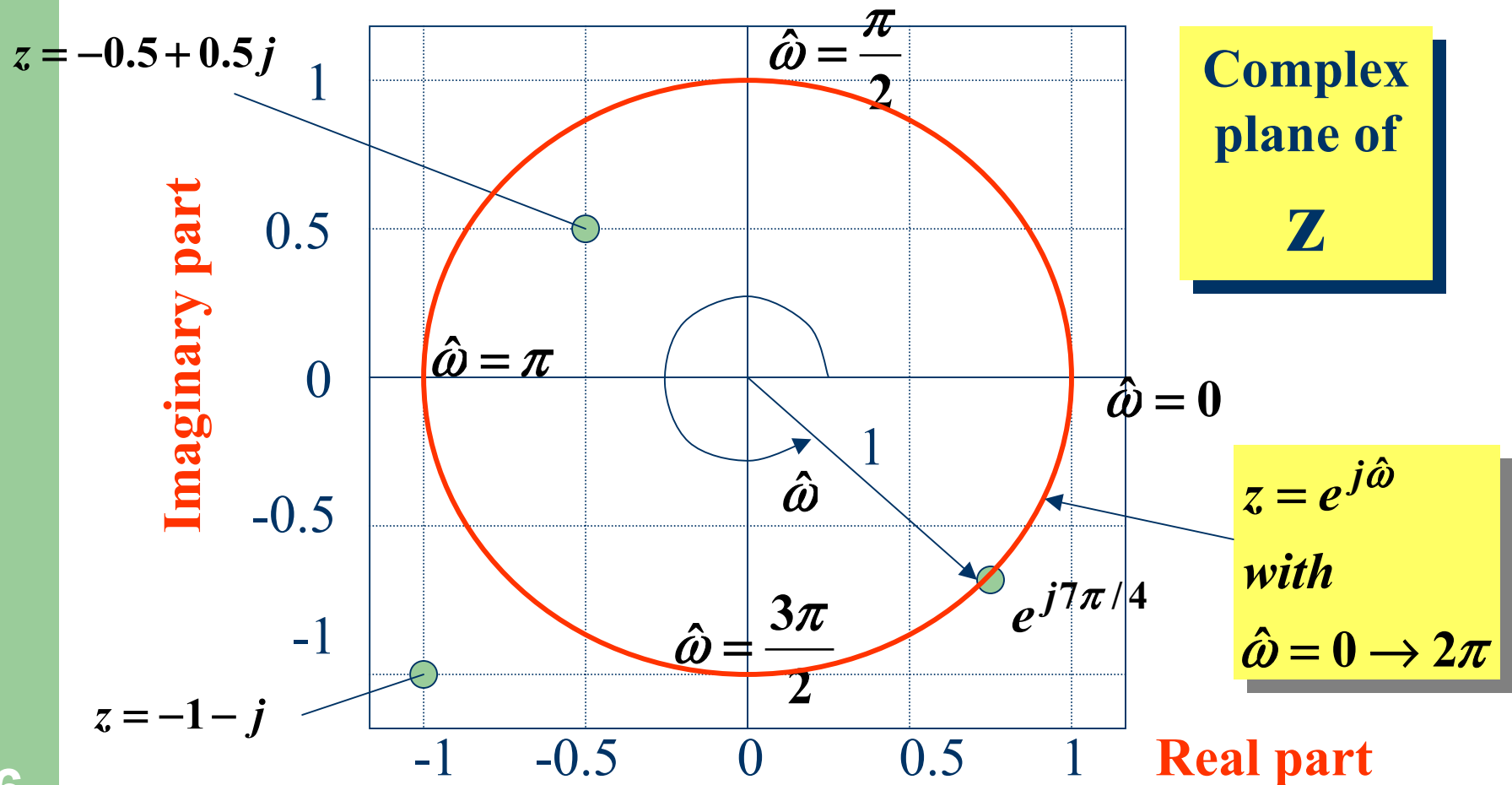
**Z-Domain**

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$$

$$H_p(\hat{\omega}) = H_p\left(e^{j\hat{\omega}}\right) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$



# Unit Circle in Complex Plane





- If we consider  $z$  is a complex number, Fourier domain lies on a special part of the Z-transform  
**⇒ The Unit Circle**
- Many properties of the frequency responses are evident from plots of system function properties in the  $z$ -plane
- For instance, it is obvious that the Fourier spectrum of discrete-time sequence is periodic since it is just going around a circle in  $z$ -plane



## Zeros and Poles of $H(z)$

- A polynomial can be characterized by its zeros and poles up to a multiplicative factor
- It is also applied to Z-transform
- For example,

$$\begin{aligned} H(z) &= 1 - 2z^{-1} + 2z^{-2} - z^{-3} \\ &= \frac{z^3 - 2z^2 + 2z - 1}{z^3} \\ &= \frac{(z-1)(z - e^{j\pi/3})(z - e^{-j\pi/3})}{z^3} \end{aligned}$$

**zeros:** the value of  $z$  such that  $H(z) = 0$

$$\Rightarrow z = 1, e^{j\pi/3}, \text{ and } e^{-j\pi/3}$$

**poles:** the value of  $z$  such that  $H(z) = \infty$

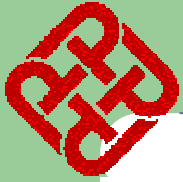
$$\Rightarrow z = 0$$

Since  $z$  is of order 3,  
there are 3 poles at  $z=0$

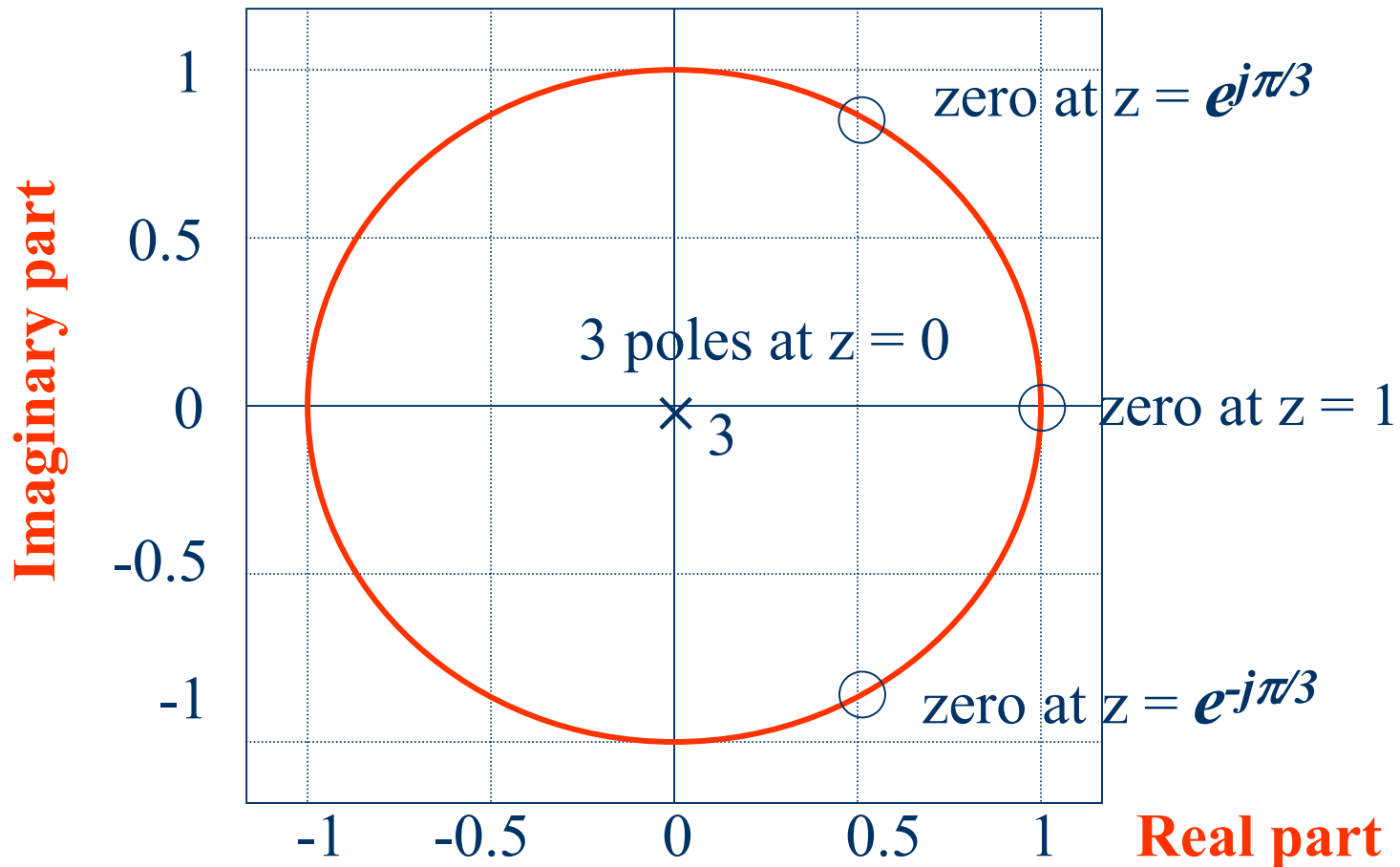


- In the example above,  $H(z)$  is characterized by its zeros at  $z = 1$ ,  $e^{j\pi/3}$ , and  $e^{-j\pi/3}$  and 3 poles at  $z = 0$
- If there is another polynomial having the same zeros and poles, it will be different from  $H(z)$  only by a scaling factor

$$\begin{aligned} H'(z) &= 0.5 - z^{-1} + z^{-2} - 0.5z^{-3} \\ &= \frac{1}{2} \frac{(z-1)(z-e^{j\pi/3})(z-e^{-j\pi/3})}{z^3} \\ &= \frac{1}{2} H(z) \end{aligned}$$

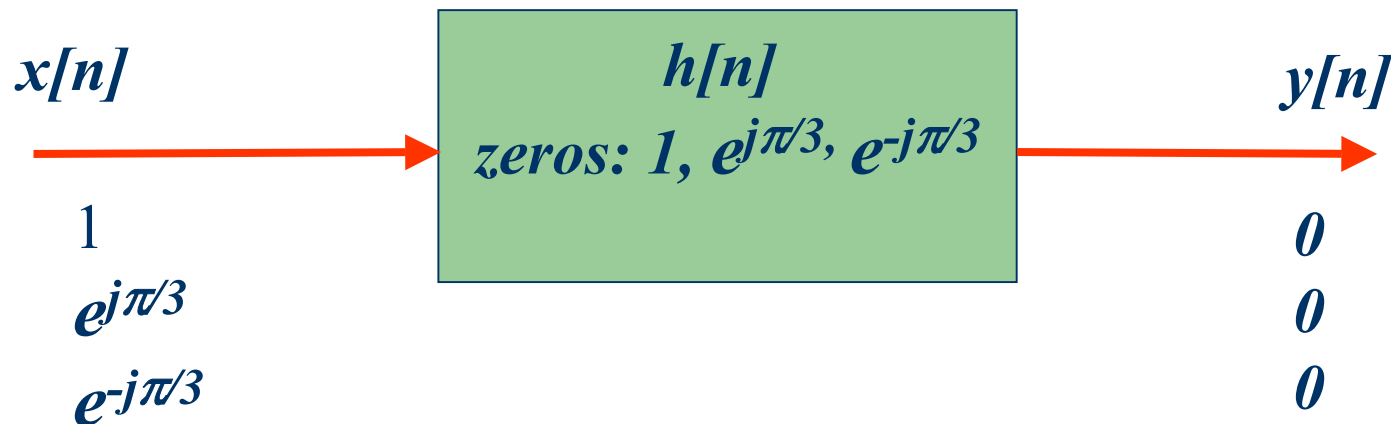


# Pole-Zero Plot





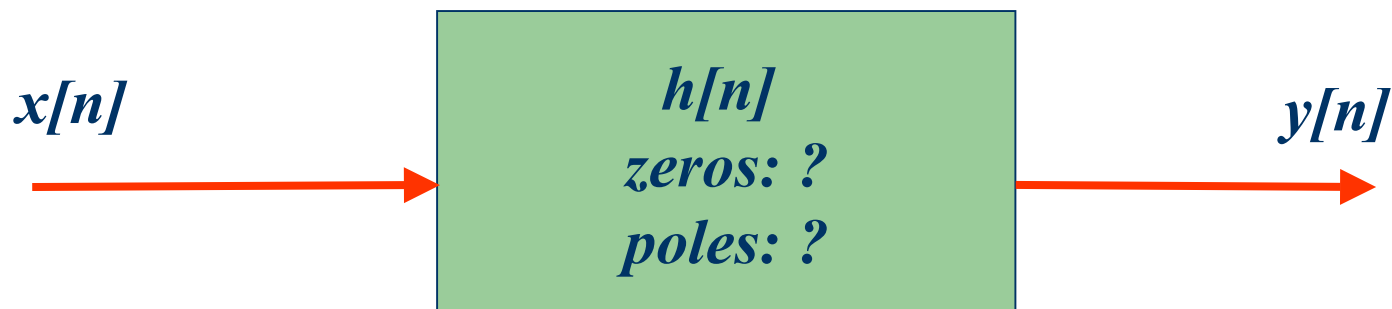
- Let's consider those **zeros on unit circle**
- Anything happens on the unit circle directly affects the frequency response of the system
- If a system has zeros as above, it means that it will reject complex sinusoid  $1, e^{j\pi/3}, e^{-j\pi/3}$






## Null Filter

- A **null filter** can selectively reject some of the frequencies in a signal
- **Example:** A signal sampled at 1000Hz is composed of two sinusoids of frequencies 100Hz and 50Hz. The signal is sent to an LTI system. Design the system such that it will reject the 50Hz component





$$\begin{aligned} X_p(\hat{\omega}) &= \frac{e^{j2\pi 100/1000} + e^{-j2\pi 100/1000}}{2} \\ &\quad + \frac{e^{j2\pi 50/1000} + e^{-j2\pi 50/1000}}{2} \\ &= \frac{e^{j\pi/5} + e^{-j\pi/5}}{2} + \boxed{\frac{e^{j\pi/10} + e^{-j\pi/10}}{2}} \end{aligned}$$

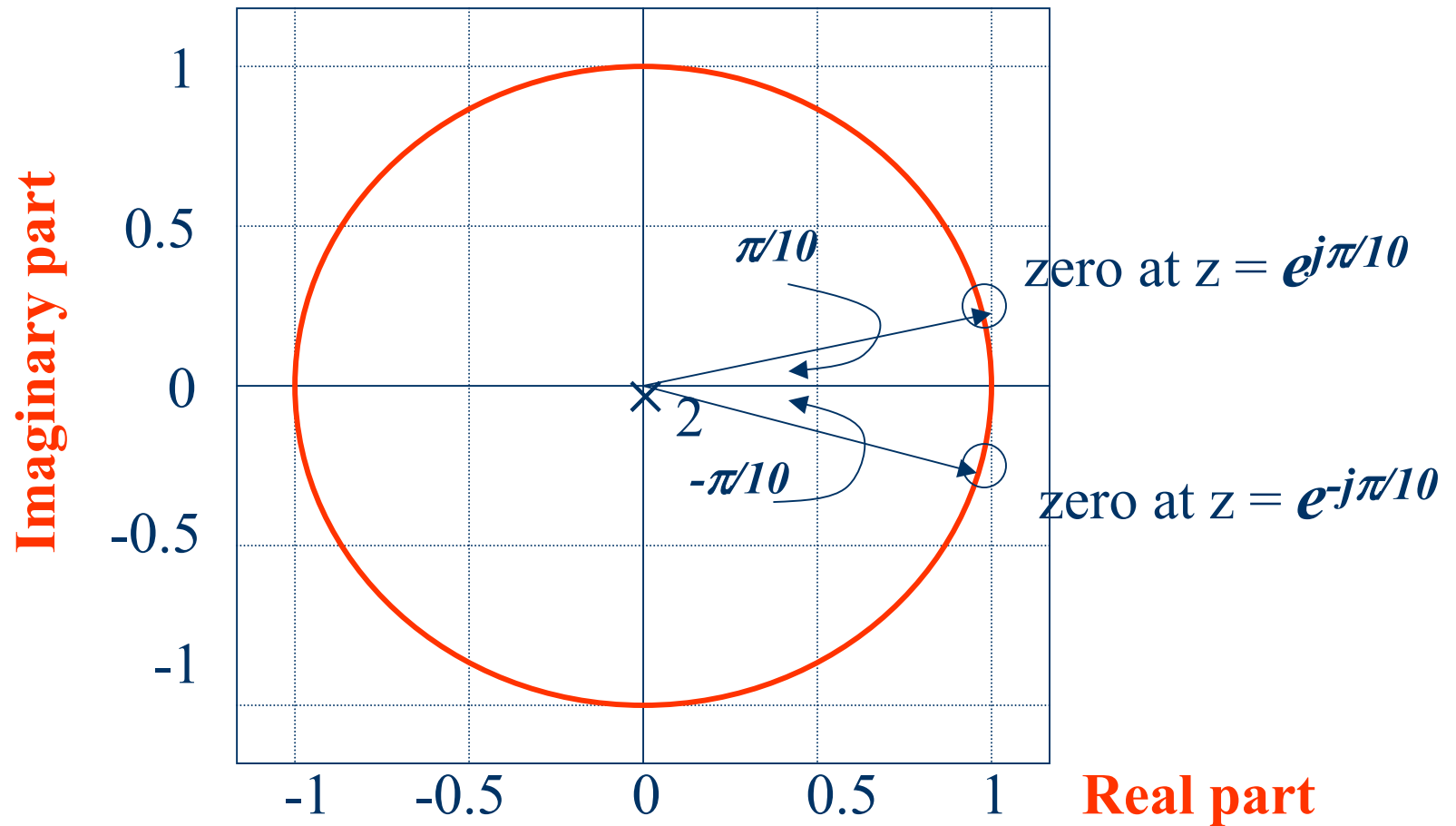
to be rejected 

- We need  $h[n]$  to have 1 zero at  $z = e^{j\pi/10}$  and another zero at  $z = e^{-j\pi/10}$



Signal Processing Fundamentals – Part I  
Spectrum Analysis and Filtering

7. Convolution, FIR Filters and Z-Transform

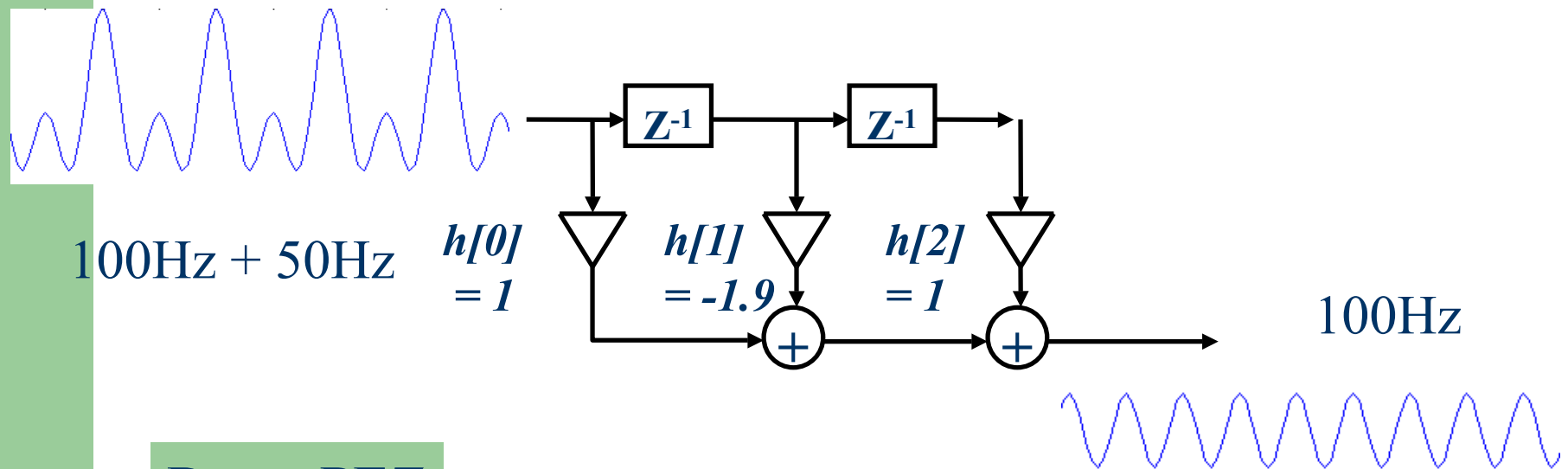




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$$\begin{aligned} H(z) &= \frac{(z - e^{j\pi/10})(z - e^{-j\pi/10})}{z^2} \\ &= 1 - 2\cos(\pi/10)z^{-1} + z^{-2} \\ &= 1 - 1.9z^{-1} + z^{-2} \end{aligned}$$



Demo PEZ



## Exercise

**Modify the null filter above such that it will reject 100Hz signal rather than 50Hz signal**



- Let us have a closer look to the running average filter as discussed before

$$h[n] = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$$

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

Geometric Series

$$= \sum_{n=0}^5 z^{-n} = \frac{1 - z^{-6}}{1 - z^{-1}} = \frac{z^6 - 1}{z^5(z - 1)}$$

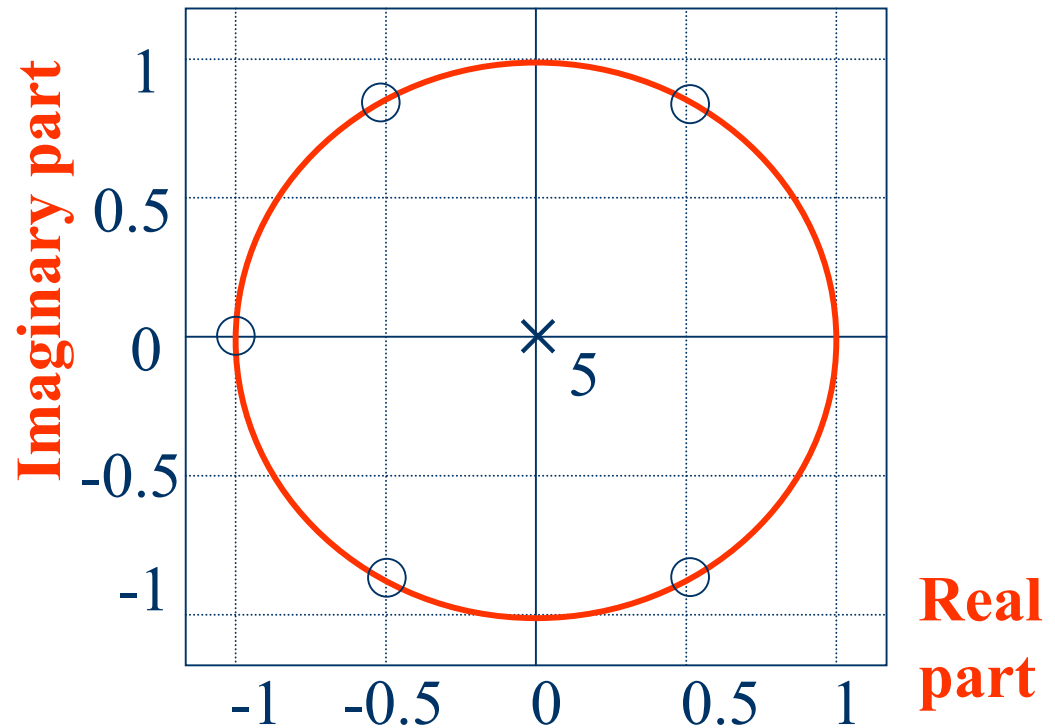
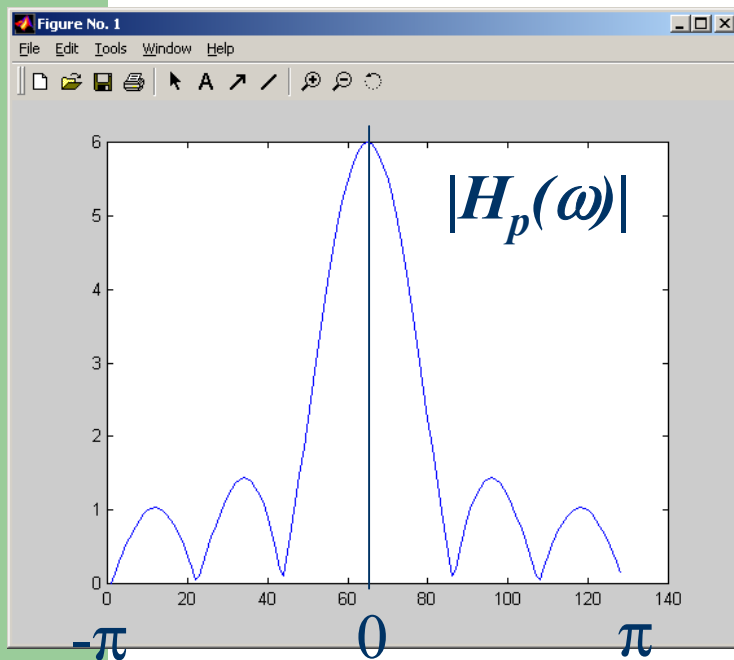
$$= \frac{(z - 1)(z - e^{j2\pi/6})(z - e^{j2\pi2/6}) \dots (z - e^{j2\pi5/6})}{z^5(z - 1)}$$

$$= \frac{(z - e^{j2\pi/6})(z - e^{j2\pi2/6}) \dots (z - e^{j2\pi5/6})}{z^5}$$



Signal Processing Fundamentals – Part I  
Spectrum Analysis and Filtering

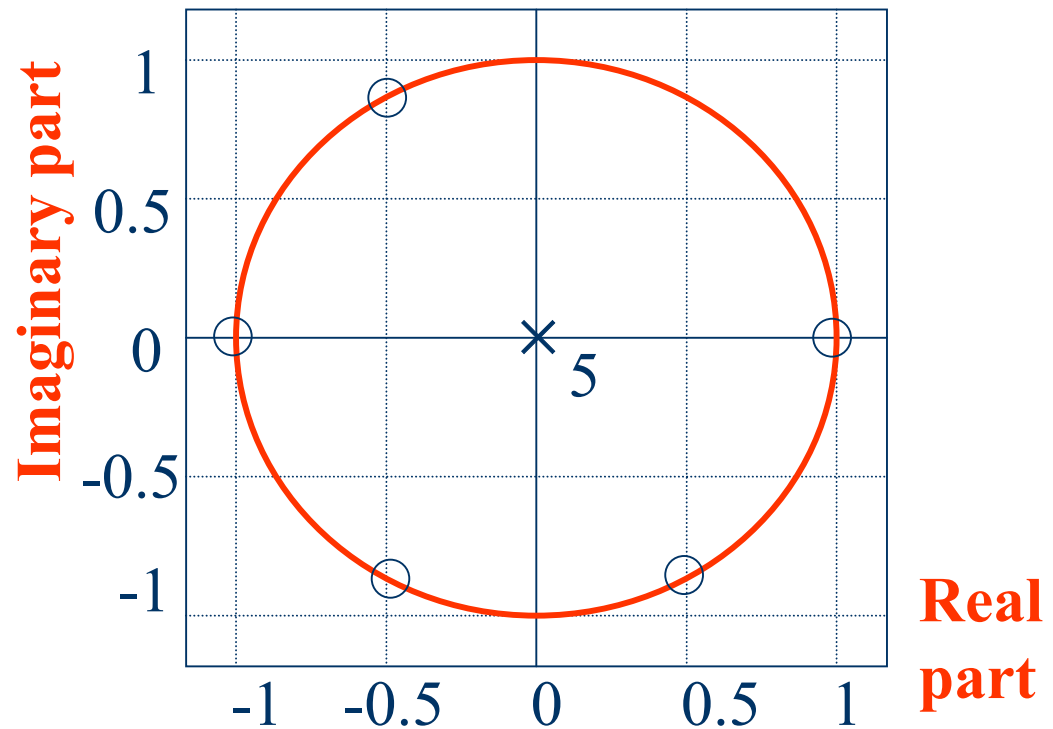
7. Convolution, FIR Filters and Z-Transform



- For every zero, it corresponds to a minimum point in the frequency domain
- As there is no zero at  $z = 1$  ( $\omega = 0$ ), it creates a gap between  $2\pi/6$  and  $-2\pi/6$ . That's why it is **low pass filter**



- It gives us a clue of how to design band pass filter  
⇒ filter that pass a particular frequency band
- For example, if we want a band pass filter that will pass frequencies centered at  $2\pi/6$ ,  $H(z)$  should have pole-zero plot as follows:





- Unfortunately, such pole-zero plot cannot be implemented by FIR filter since the coefficients would be complex numbers

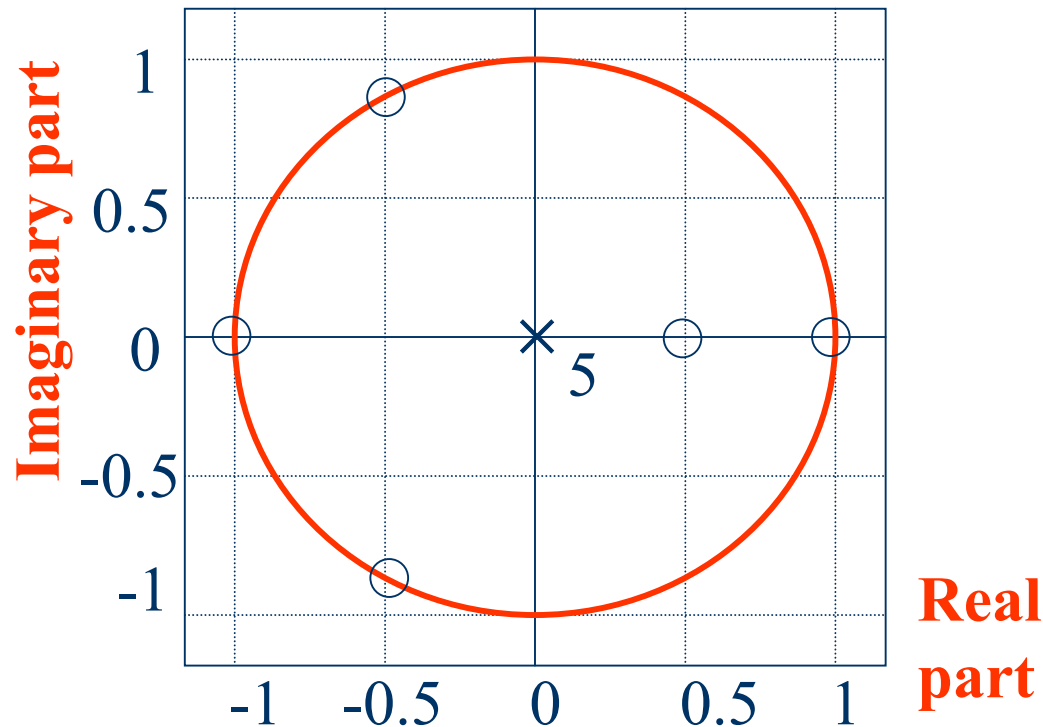
$$\begin{aligned} H(z) &= \frac{(z-1)(z-e^{j2\pi2/6})(z-e^{j2\pi3/6})(z-e^{j2\pi4/6})(z-e^{j2\pi5/6})}{z^5} \\ &= \frac{(z-1)(z-e^{j2\pi3/6})(z-e^{j2\pi2/6})(z-e^{j2\pi4/6})(z-e^{j2\pi5/6})}{z^5} \\ &= \frac{(z-1)(z+1)(z-e^{j2\pi2/6})(z-e^{-j2\pi2/6})(z-e^{j2\pi5/6})}{z^5} \\ &= \frac{(z^2-1)(z^2-2\cos(2\pi2/6)+1)(z-e^{j2\pi5/6})}{z^5} \end{aligned}$$

They are all  
real numbers

must be  
complex



- To have real coefficients, a zero must be paired with its complex conjugate
- Let's consider the following pole-zero plot





- We can easily implement such pole-zero plot as follows:

$$H(z) = \sum_{n=0}^5 (\cos(2\pi n / 6)) z^{-n}$$

$$\begin{aligned} H(z) = & 1 + \cos(2\pi / 6) z^{-1} + \cos(2\pi 2 / 6) z^{-2} \\ & + \cos(2\pi 3 / 6) z^{-3} + \cos(2\pi 4 / 6) z^{-4} \\ & + \cos(2\pi 5 / 6) z^{-5} \end{aligned}$$



## Proof

$$\begin{aligned} H(z) &= \sum_{n=0}^5 (\cos(2\pi n / 6)) z^{-n} \\ &= \frac{1}{2} \left\{ \sum_{n=0}^5 e^{j2\pi n / 6} z^{-n} + \sum_{n=0}^5 e^{-j2\pi n / 6} z^{-n} \right\} \end{aligned}$$

Since it is known from the geometric series that

$$\sum_{n=0}^{L-1} x^{-n} = \frac{1 - x^{-L}}{1 - x^{-1}} = \frac{x^L - 1}{x^{L-1}(x - 1)}$$



Hence the above equation can be written as

$$H(z) = \frac{1}{2} \left\{ \frac{z^6 - 1}{z^5(z - p)} + \frac{z^6 - 1}{z^5(z - p^*)} \right\}$$

where  $p = e^{j2\pi/6}$ ,  $p^*$  is the complex conjugate of  $p$

$$\begin{aligned} H(z) &= \frac{1}{2} \left\{ \frac{(z^6 - 1)(z - p^*) + (z^6 - 1)(z - p)}{z^5(z - p)(z - p^*)} \right\} \\ &= \frac{(z^6 - 1) \left( z - \frac{1}{2}(p + p^*) \right)}{z^5(z - p)(z - p^*)} = \frac{(z^6 - 1)(z - \cos(2\pi/6))}{z^5(z - e^{j2\pi/6})(z - e^{-j2\pi/6})} \end{aligned}$$



It is known that

$$z^6 - 1 = (z - 1)(z - e^{j2\pi/6})(z - e^{j2\pi2/6}) \dots (z - e^{j2\pi5/6})$$

Hence

$$\begin{aligned} H(z) &= \frac{(z^6 - 1)(z - \cos(2\pi/6))}{z^5 (z - e^{j2\pi/6})(z - e^{-j2\pi/6})} \\ &= \frac{(z - 1)(z - e^{j2\pi/6}) \dots (z - e^{j2\pi5/6})(z - \cos(2\pi/6))}{z^5 (z - e^{j2\pi/6})(z - e^{-j2\pi/6})} \\ &= \frac{(z - 1)(z - e^{j2\pi2/6}) \dots (z - e^{j2\pi4/6})(z - \cos(2\pi/6))}{z^5} \end{aligned}$$



## Exercise

If we want to have the running average filter that can give a pass band centered at  $2\pi/6$ , what is the filter coefficients and the associated pole-zero plot?