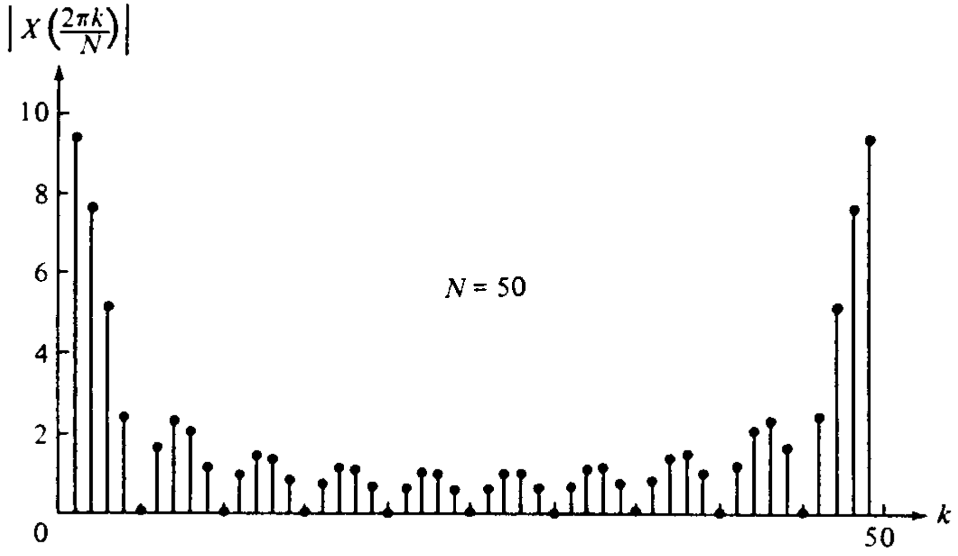
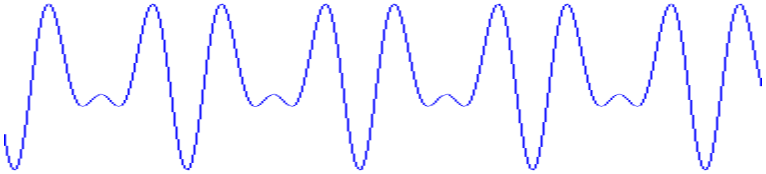


5. Fourier Transform and Spectrum Analysis

Spectrum of Non-periodic Signals

- Fourier series help us to find the spectrum of periodic signals



- Most signals are not periodic
  - Speech, audio, etc.

- Need another tool to find the spectrum of non-periodic (aperiodic) signals

  ⇒ Fourier Transform

Fourier Transform of Discrete-time Signals

- **Let *x(t)* be an aperiodic continuous-time signal, *x[n]* is the samples of *x(t)* such that:**

$$x[n] = x(nT_s)$$

- **The spectrum of *x[n]* is given by:**

Radian frequency

$$\hat{\omega} = \omega T_s$$

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n T_s} \quad \text{or} \quad X_p(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega} n}$$

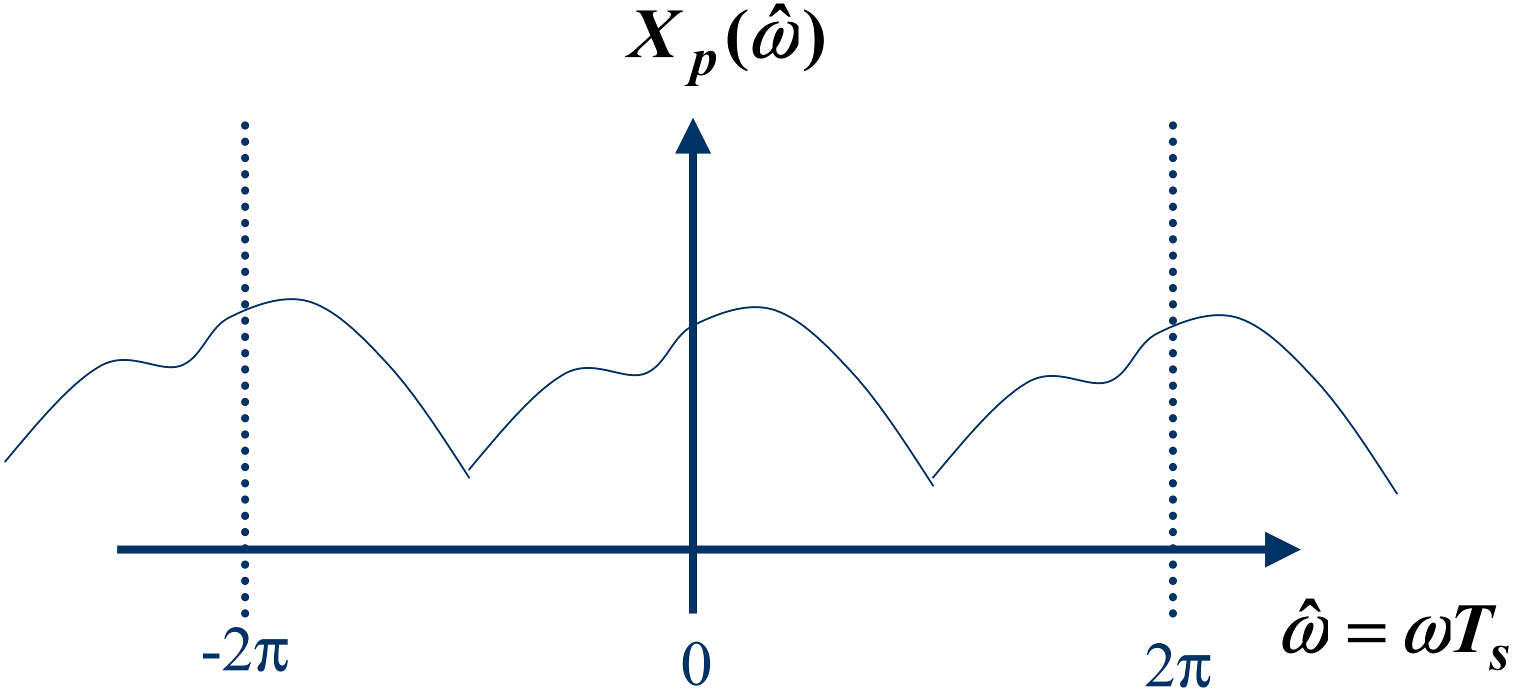
Aperiodic Signals have Periodic Spectrum

- **It is interesting to note that $X_p(\omega)$ is periodic since**

$$X_p\left(\hat{\omega}+2\pi k\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}+2\pi k)n}$$

$$=\sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}\cancelto{1}{e^{-j2\pi nk}}=X_p(\hat{\omega})$$

5. Fourier Transform and Spectrum Analysis



5. Fourier Transform and Spectrum Analysis

- If $x(t)$ has a spectrum of $X(\omega)$ and $x[n] = x(nT_s)$ has a spectrum of $X_p(\omega)$, it can be shown that

$$X_p(\hat{\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\hat{\omega} + 2\pi k) \qquad -\infty \leq \omega \leq \infty$$

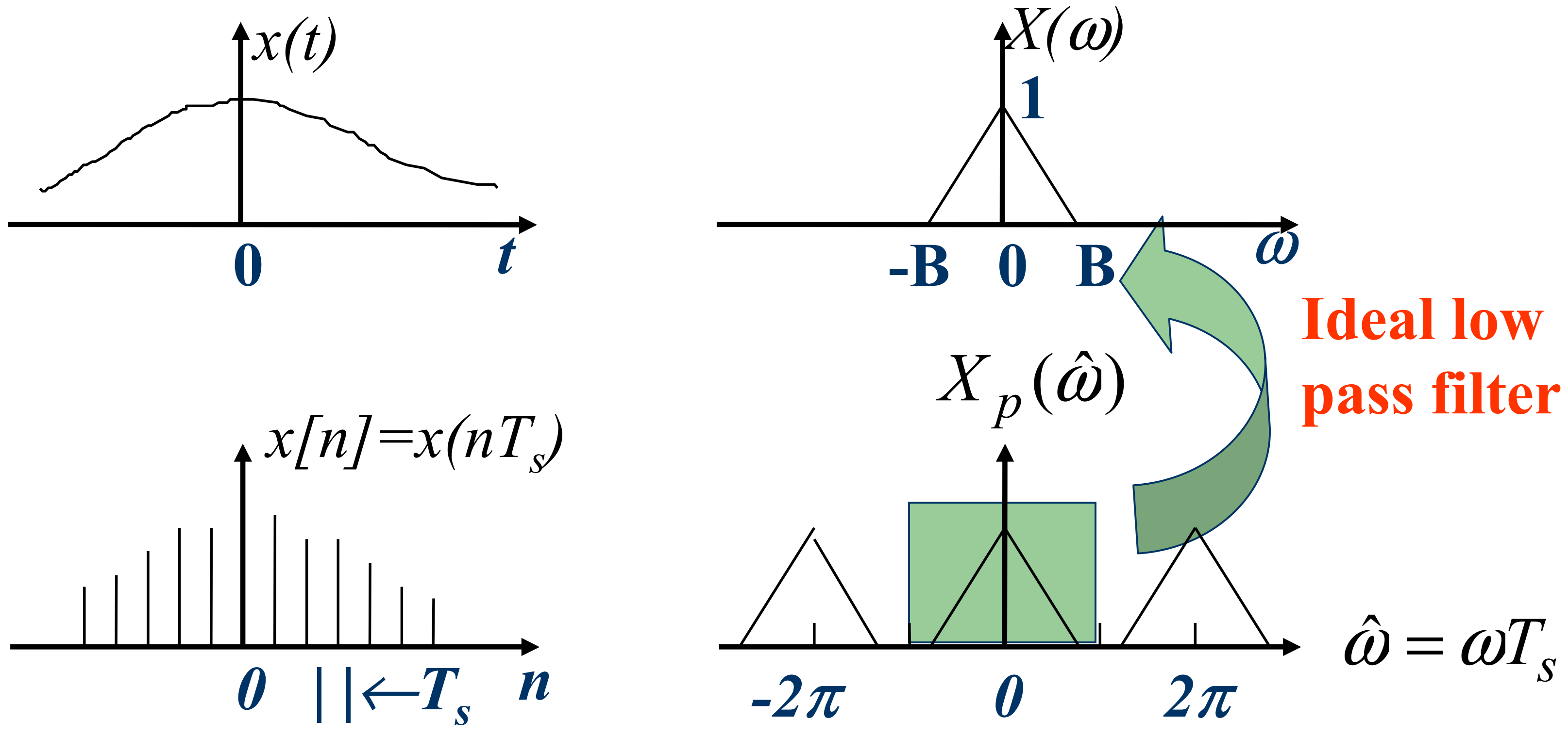
$$= \ldots + \frac{1}{T_s} X(\hat{\omega} - 2\pi)$$

$$+ \frac{1}{T_s} X(\hat{\omega})$$

$$+ \frac{1}{T_s} X(\hat{\omega} + 2\pi) + \ldots$$

Time Domain

Frequency Domain

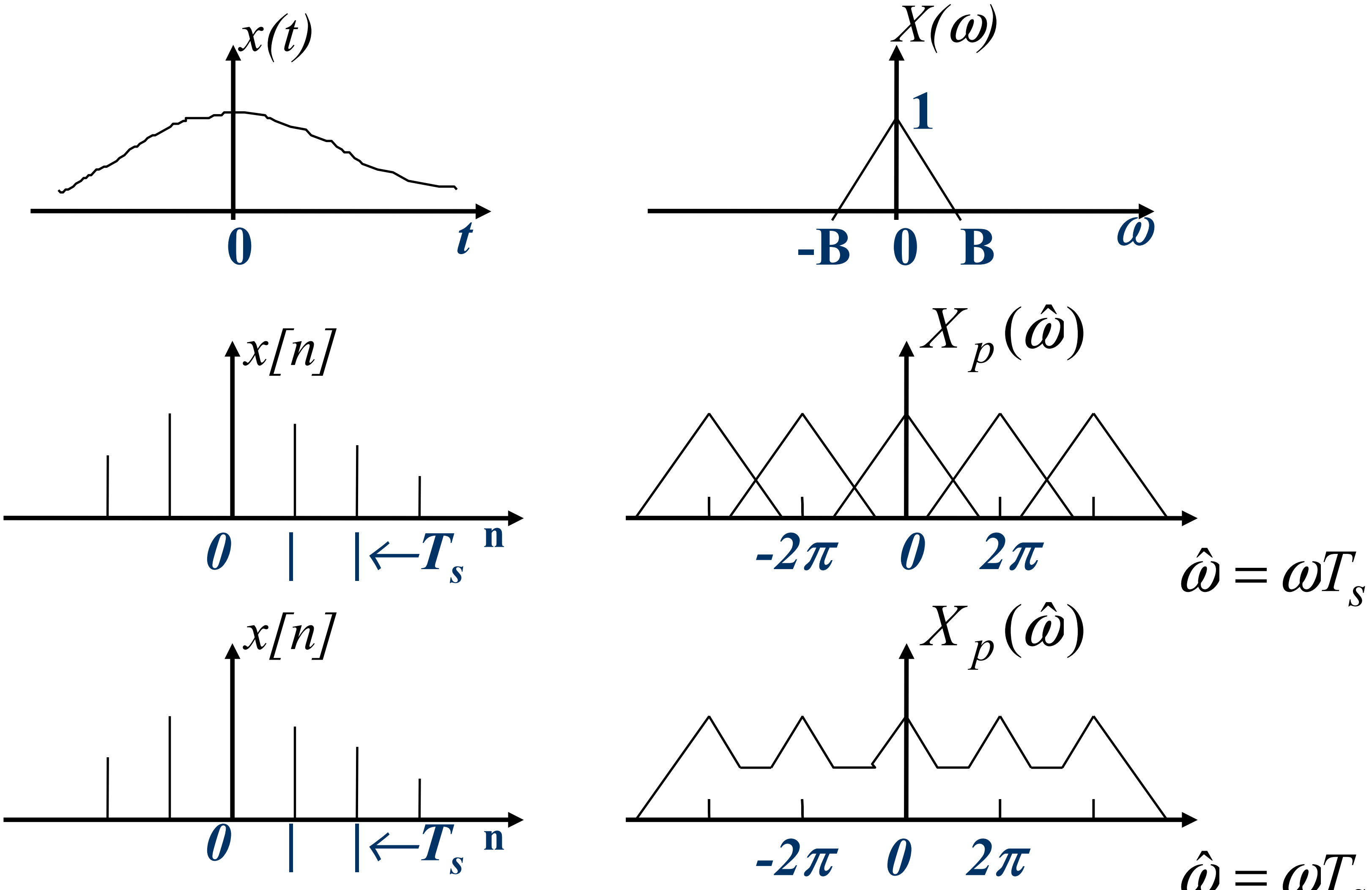


Ideal low pass filter

5. Fourier Transform and Spectrum Analysis

Time Domain

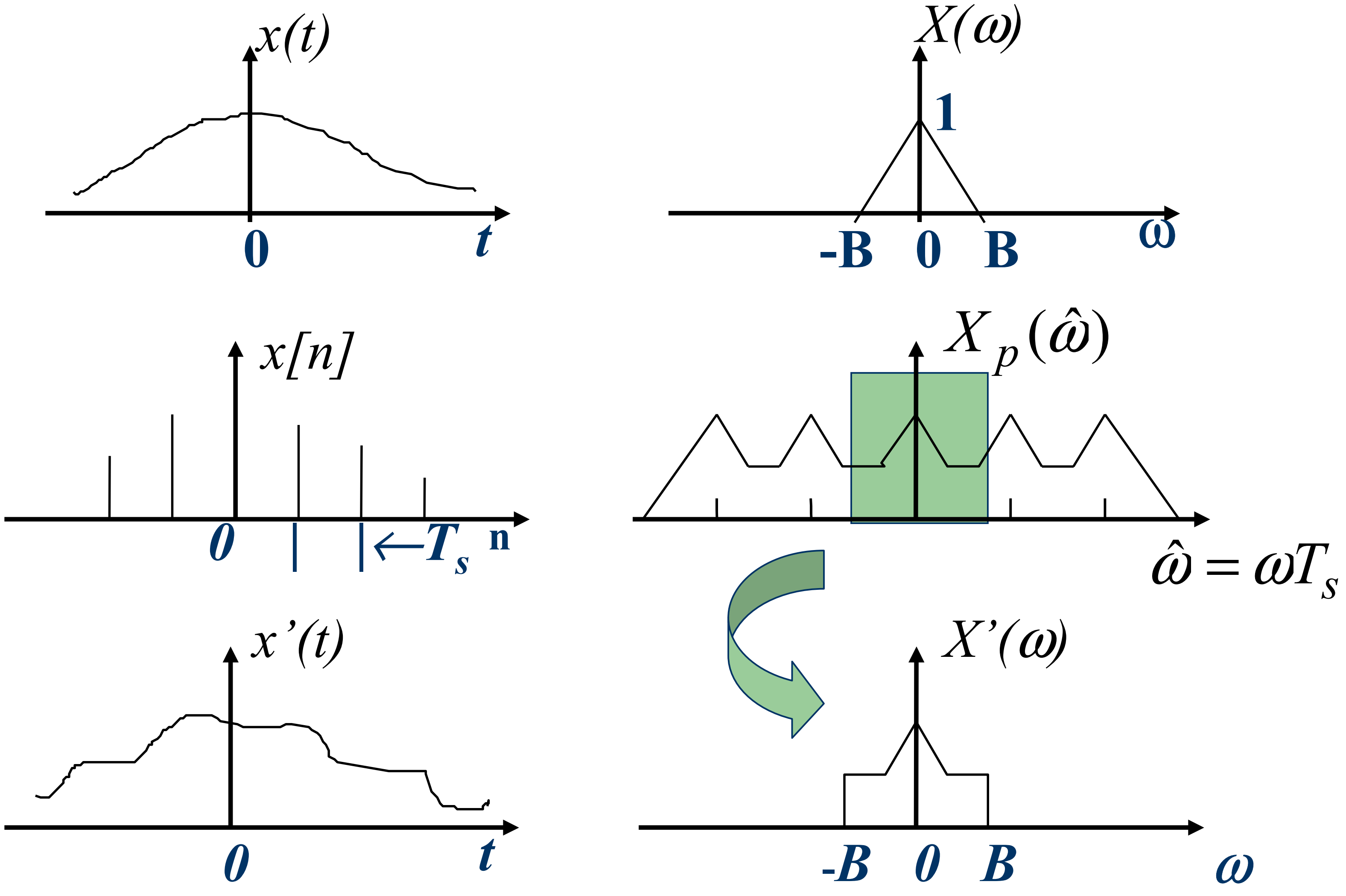
Frequency Domain



5. Fourier Transform and Spectrum Analysis

Time Domain

Frequency Domain



- **If the signal has frequency components beyond $|\pi|$, after sampling, these frequency components will affect the other replicas in the spectrum**
- **Even with an ideal low pass filter, the original signal cannot be reconstructed. This is the so-called alias effect**
- **Restate the Shannon Sampling Theorem for general aperiodic signals**

$$\hat{\omega} \le |\pi| \Rightarrow 2\pi f_{\max} T_s \le |\pi|$$

$$\Rightarrow 2\pi f_{\max} \le |\pi f_s| \quad or \quad f_s \ge 2 f_{\max}$$

Shannon Sampling Theorem

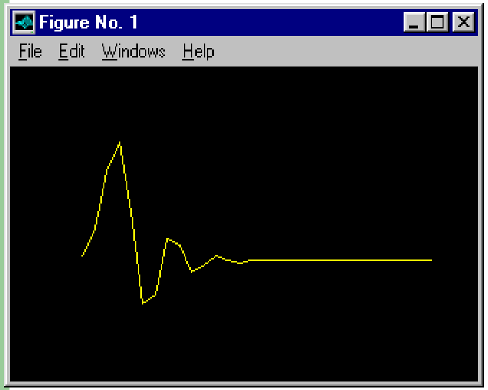
A continuous-time aperiodic signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$ if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$

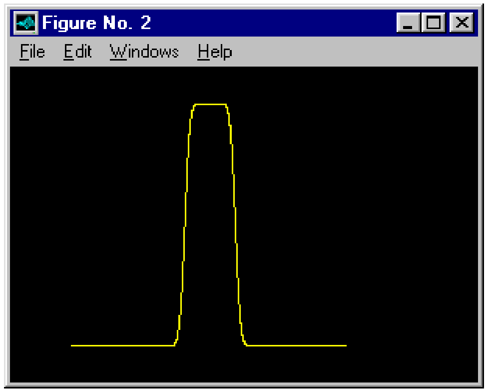
Nyquist Frequency

Real Examples

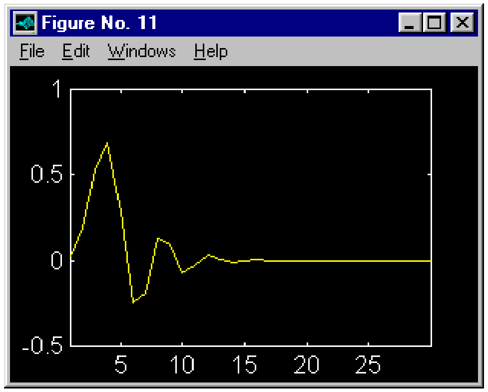
Time Domain

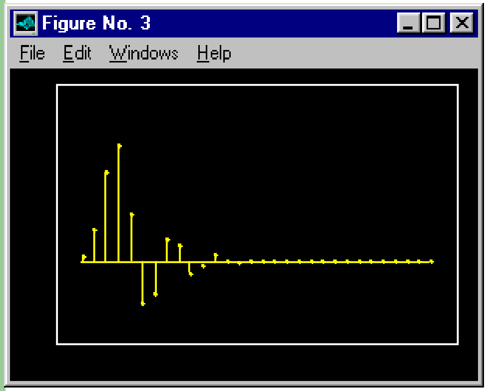
Frequency Domain

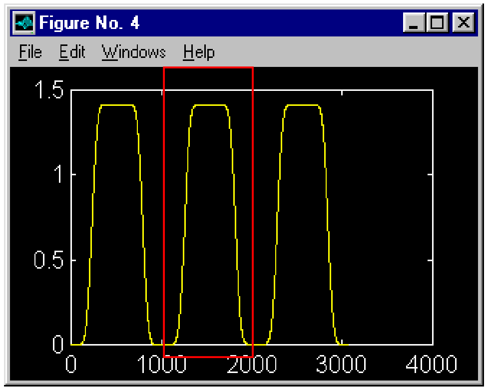




Resulted signal





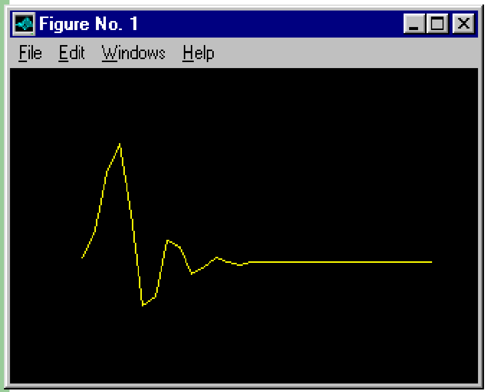


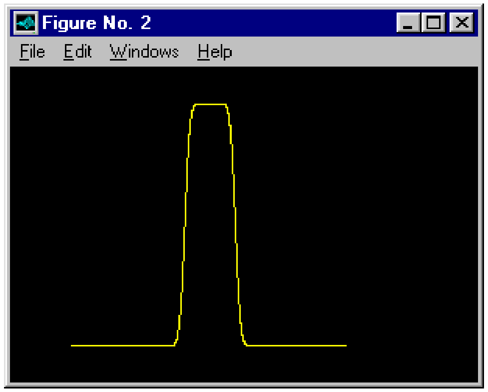
Ideal low pass filter

5. Fourier Transform and Spectrum Analysis

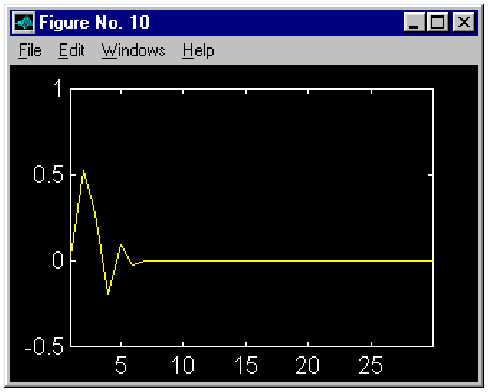
Time Domain

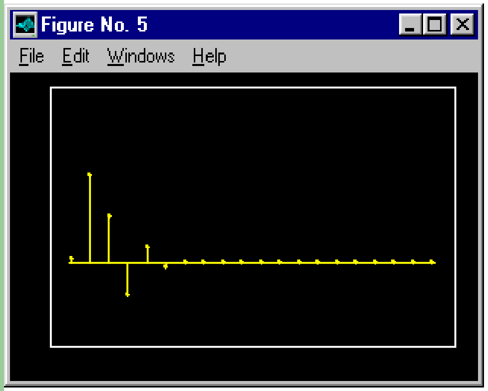
Frequency Domain

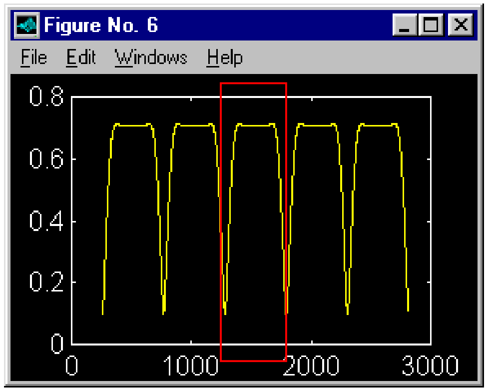




Resulted signal





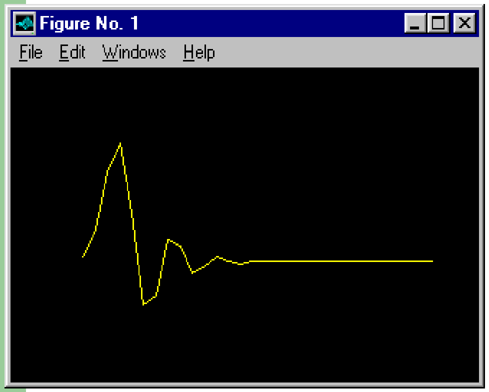


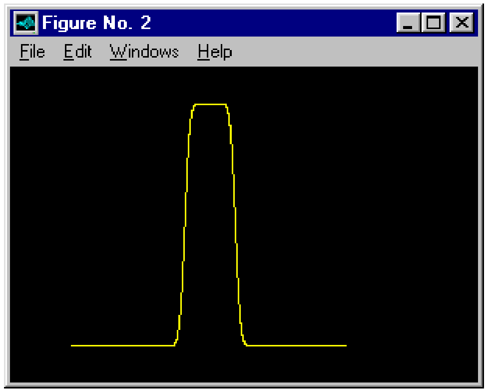
Ideal low pass filter

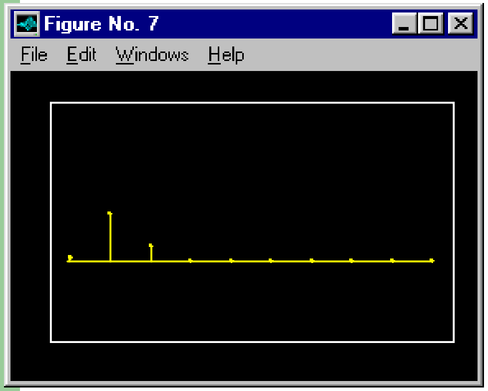
5. Fourier Transform and Spectrum Analysis

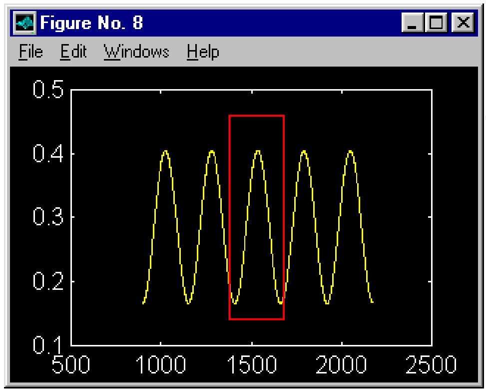
Time Domain

Frequency Domain

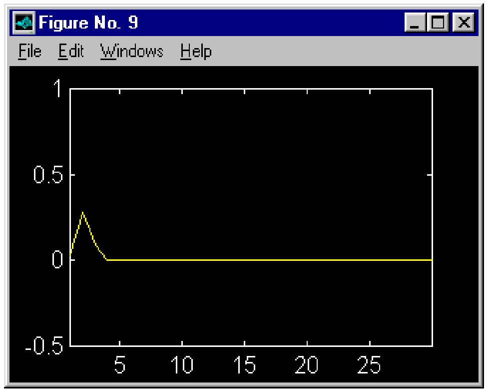








Resulted signal



Ideal low pass filter

How to Solve Aliasing Problems?

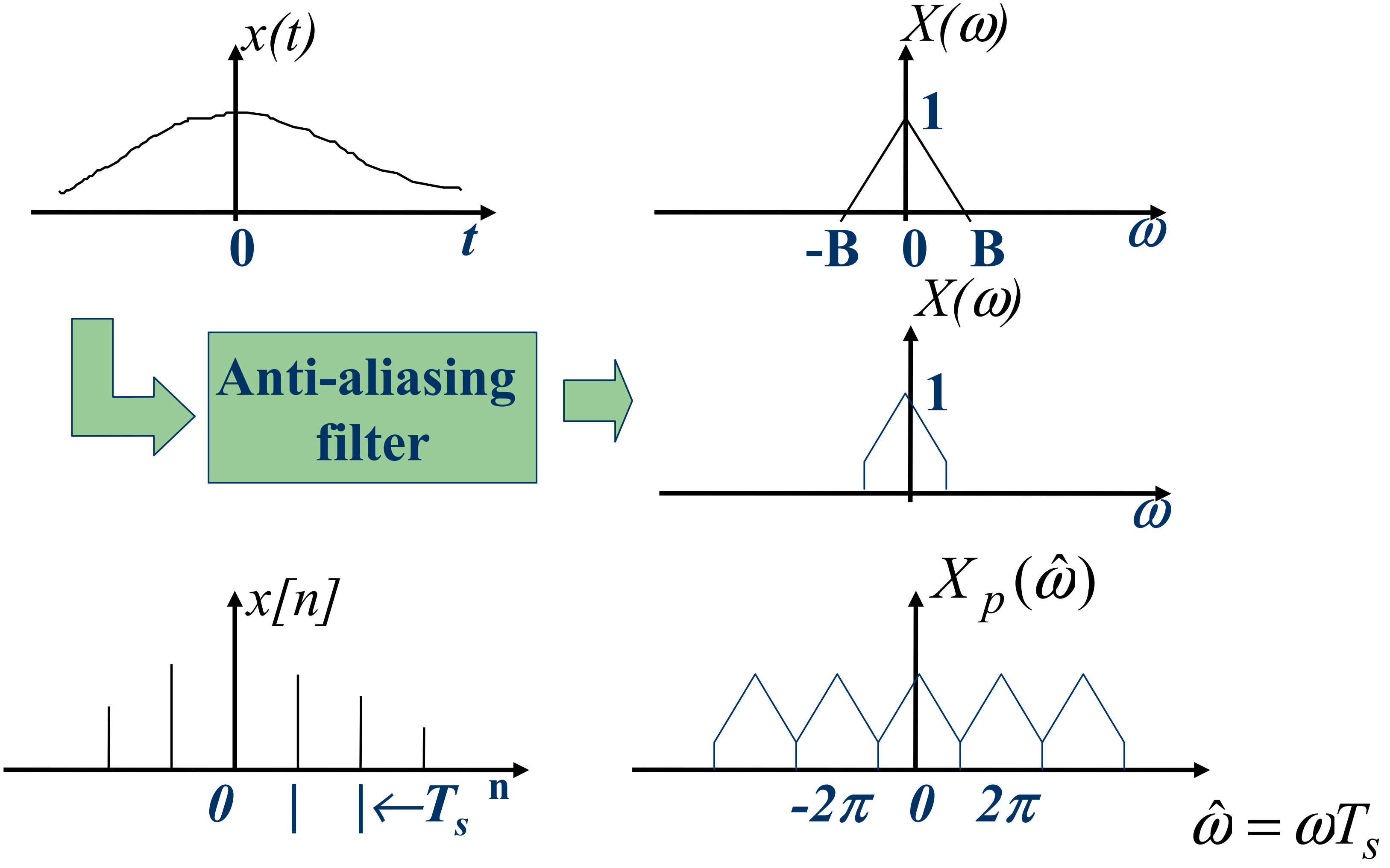
1. Increase the sampling rate such that $f_s \geq 2f_{max}$
2. Use anti-aliasing filter first

Pre-filter the input signal such that it will never has frequency components beyond |π|

Pre-filter → A/D → Digital Signal Processor → D/A → Post-filter

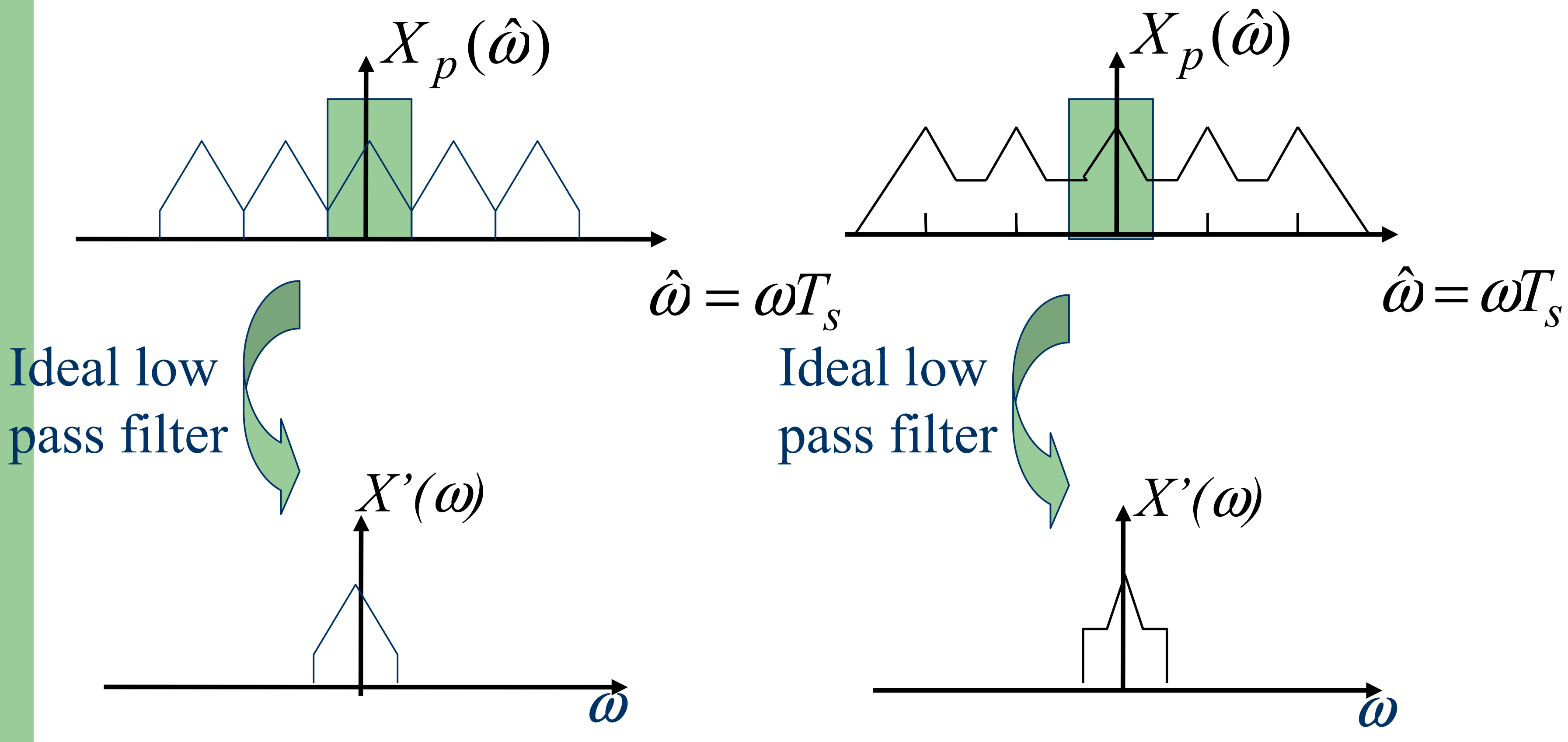
5. Fourier Transform and Spectrum Analysis

Time Domain **Frequency Domain**



With anti-aliasing filter

Without anti-aliasing filter



Ideal low pass filter

Ideal low pass filter

Look better

Hear the effect!

Original 44.1kHz sampling

8kHz sampling with aliasing

8kHz sampling with anti-aliasing filter

Discrete Fourier Transform

- **Spectrum of aperiodic discrete-time signals is periodic and continuous**
- **Difficult to be handled by computer**
- **Since the spectrum is periodic, there's no point to keep all periods – one period is enough**
- **Computer cannot handle continuous data, we can only keep some samples of the spectrum**
- **Interesting enough, such requirements lead to a very simple way to find the spectrum of signals**

⇒ Discrete Fourier Transform

5. Fourier Transform and Spectrum Analysis

- **Recall the Fourier transform of an aperiodic discrete sequence**

$$X_p(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

- **Assume *x[n]* is an aperiodic sequence with *N* values, i.e. *{x[n] : n = 0, 1, ..., N-1}***

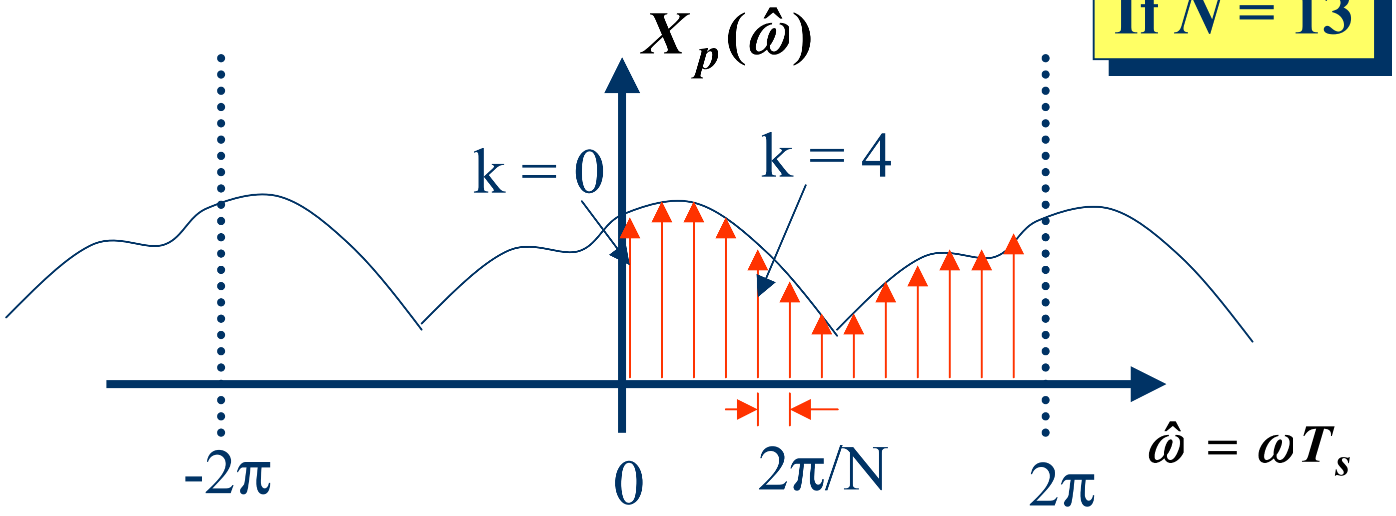
$$X_p(\hat{\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\hat{\omega}n}$$

5. Fourier Transform and Spectrum Analysis

- **If we are now interested only in *N* equally spaced frequencies of 1 period of the Fourier spectrum, i.e.**

$$X[k] = X_p\left(\frac{k.2\pi}{N}\right) \qquad k = 0,1,\ldots N-1$$

If *N* = 13



- Now if we want to compute the value of these N frequencies,

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{k.2\pi}{T_s N}\right) n T_s}$$

$$\boxed{W_N^{nk} = e^{-j2\pi nk/N}}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-jk.2\pi n/N} = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

for $k = 0,1,\ldots, N\text{-}1$

Discrete Fourier Transform

- **Discrete Fourier Transform (DFT)** is exactly the output of the Fourier Transform of an aperiodic sequence at some particular frequencies

- Inherently **periodic** since *X[k+N] = X[k]*, although we always only consider one period of *X[k]*

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$$

$$X[k+N] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi n(k+N)/N}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \cancelto{1}{e^{-j2\pi n}} = X[k]$$

- **If we know *X[k]*, we can reconstruct back the signal *x[n]* via the inverse discrete Fourier transform**

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k] e^{j2\pi nk/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} X[k] W_N^{-nk} \qquad \textit{for } n = 0,1,\ldots, N\text{-}1$$

Inverse Discrete Fourier Transform

- It can be proven as follows:

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk2\pi n/N} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[m]e^{-jk2\pi m/N}e^{jk2\pi n/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[m]e^{jk.2\pi(n-m)/N}$$

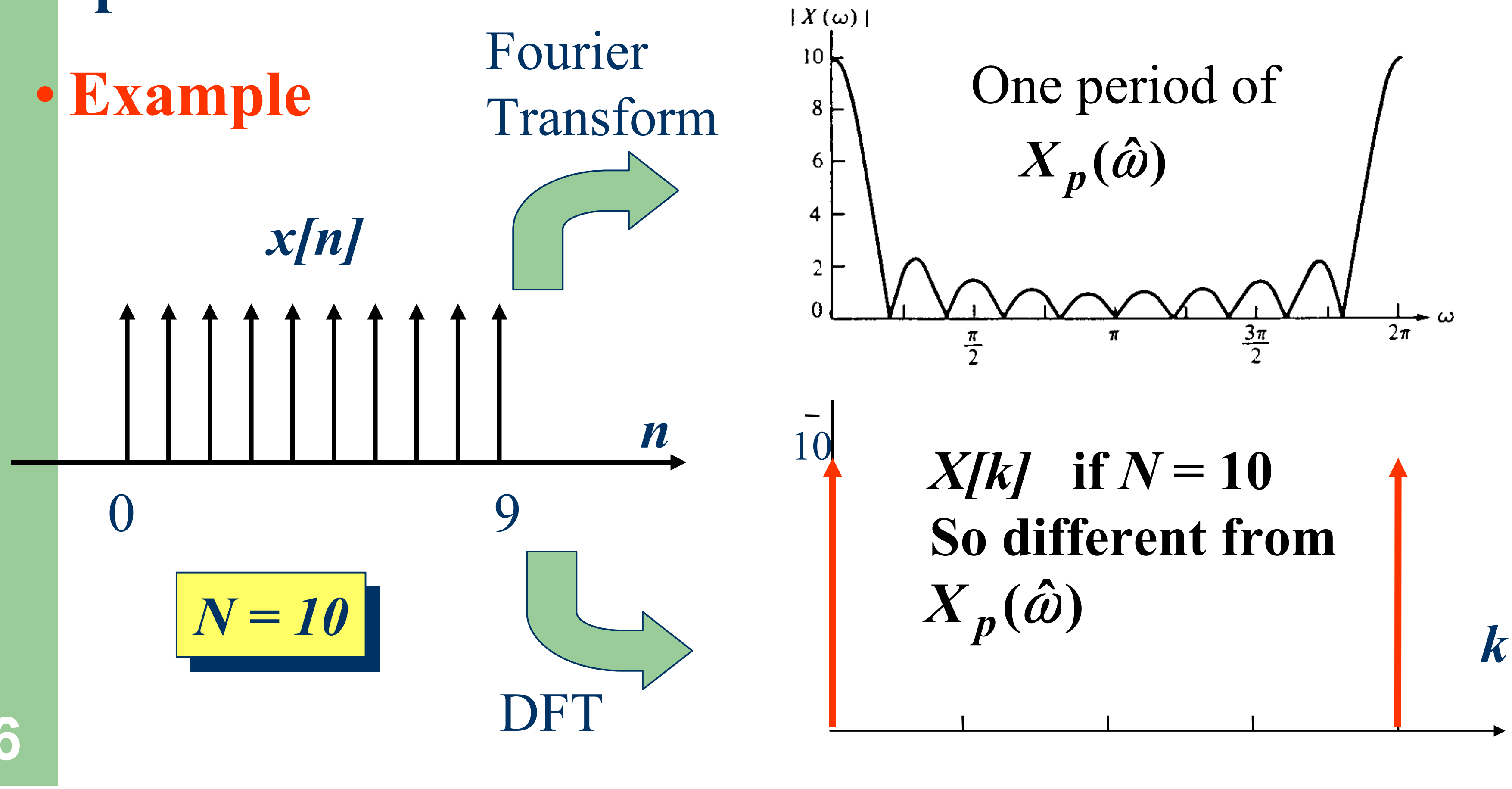
$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\left[\sum_{k=0}^{N-1} e^{jk.2\pi(n-m)/N}\right]$$

$$\begin{cases} 0 & if \quad n \neq m \\ N & otherwise \end{cases}$$

$$= x[n] \qquad for \quad n = 0,1,\ldots,N-1$$

- **Although DFT gives exact frequency response of a signal, sometimes it may not give the desired spectrum**

- **Example**



Fourier Transform

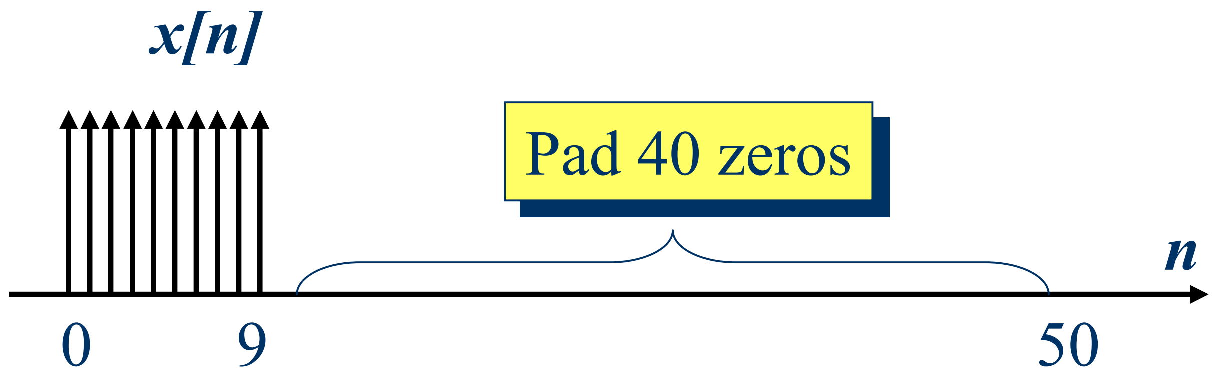
One period of $X_p(\hat{\omega})$

$N = 10$

DFT

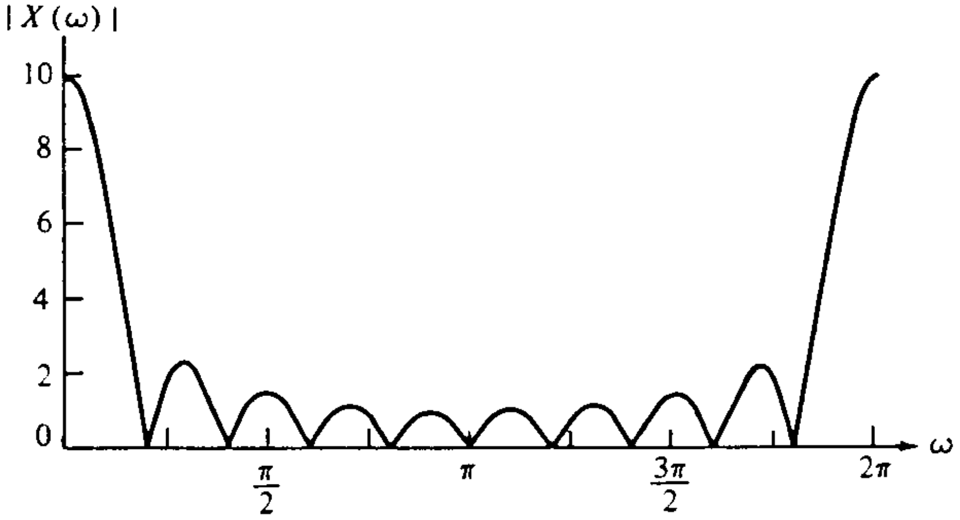
$X[k]$ if $N = 10$
So different from $X_p(\hat{\omega})$

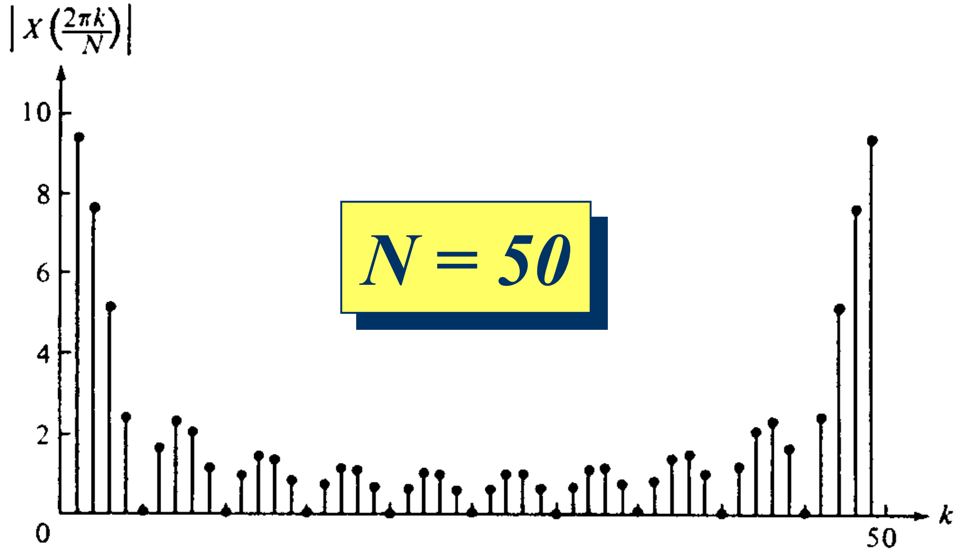
- **Need improved resolution**
- **Achieve by padding zero to the end of *x[n]* to make *N* bigger**



$$x[n] = \{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \underbrace{0\ 0\ 0\ 0\ \ldots\ 0}_{\text{40 zeros}}\}$$

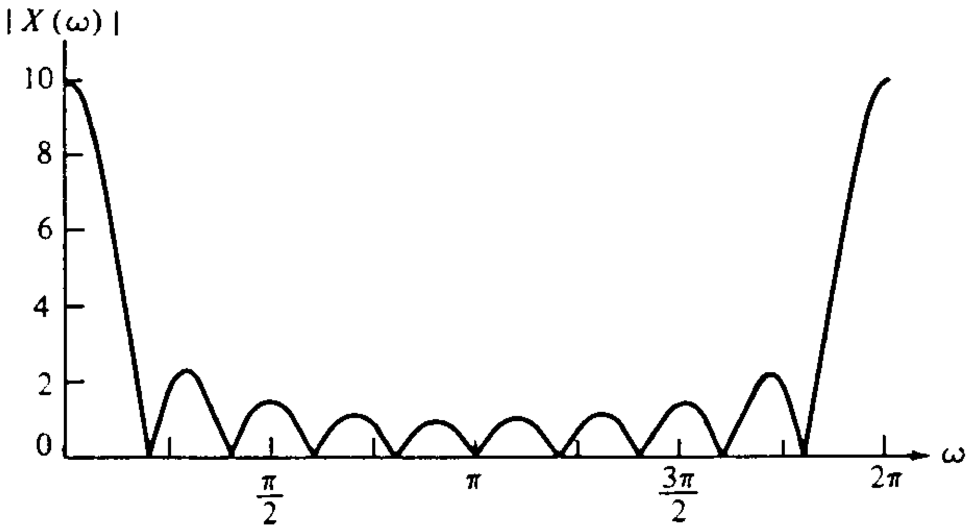
5. Fourier Transform and Spectrum Analysis

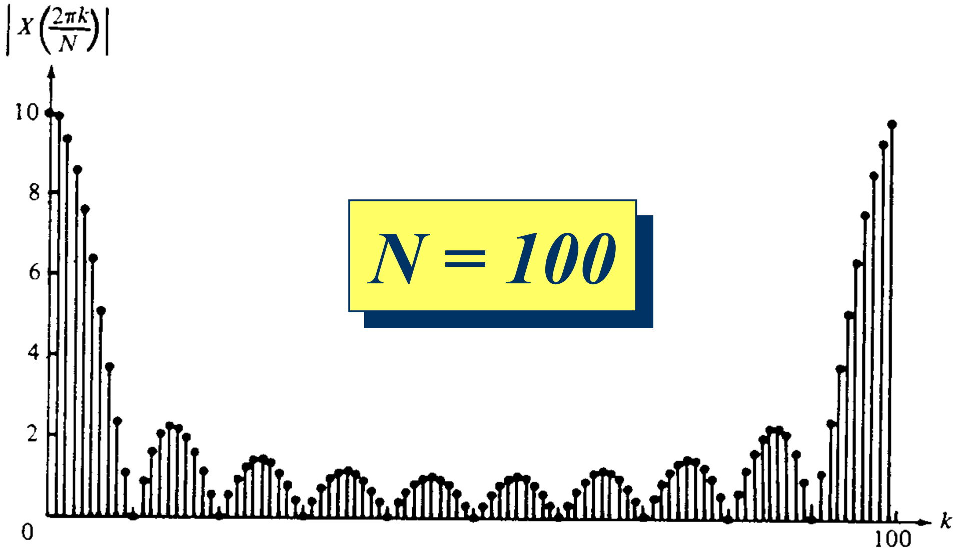




$N = 50$

5. Fourier Transform and Spectrum Analysis





$N = 100$

Exercise

Given that *x[n]* is defined in the following figure, determine its spectrum using DFT with $N = 4$