Polarization Estimation With a Dipole-Dipole Pair, a Dipole-Loop Pair, or a Loop-Loop Pair of Various Orientations

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Abstract—This work aims to estimate the polarization of fully polarized sources, given prior knowledge of the incident sources’ azimuth-elevation directions-of-arrival, using a pair of diversely polarized antennas—two electrically small dipoles, or two small loops, or one each. The pair may be collocated, or spatially separated by a known displacement. Each antenna may orient along any Cartesian coordinate. Altogether, 15 antenna/orientation configurations are thus possible. For each configuration, this paper derives 1) the closed-form polarization-estimation formulas, 2) the associated Cramér–Rao bounds, and 3) the associated computational numerical stability.

Index Terms—Antenna arrays, aperture antennas, array signal processing, parameter estimation, polarization estimation.

I. INTRODUCTION

POLARIMETRY is the measurement and the interpretation of the polarization of transverse waves [2]. The polarization state of an incident wave could reveal attributes intrinsic to the emitter or the reflector, e.g., a star’s photosphere asymmetry due to “limb polarization” [33]. Commonly used for polarization-estimation are electrically short dipoles and small loops.1 Such a dipole (loop), when oriented along a Cartesian coordinate, would measure the electric-field (magnetic-field) component of the incident transverse wavefield along that axis.

Polarization is bivariate; hence, a minimum of two diversely polarized antennas are needed for polarimetry of a fully polarized wavefield. If limiting the choice of antennas to linearly polarized antennas (i.e., dipole(s) or/and loop(s)) aligned along some Cartesian axes, there exist \( \frac{5 \times 5}{2} \) = 15 possible antenna/orientation configurations, because two components are here chosen, out of a total of six electromagnetic components. The open literature presently offers no comprehensive comparison among these 15 configurations, even though the effectiveness of polarimetry depends critically on what and how the antennas are employed. This literature gap is filled by the present work. Such dipole/dipole, loop/loop, or dipole/loop antenna-pairs have been much investigated in the open literature.

1) **One dipole and one loop**, collocated and identically oriented, are labeled a “cocentered orthogonal loop and dipole” (COLD) array. Its physical implementation and electromagnetics are discussed in [20], [24], [43], [49], [52], and [53]. It is used to estimate the arrival-angles and/or the polarization in [14], [39], [48], [57], [63], [64], [68], [76], and [78]:
   a) specifically in the vertical orientation in [14], [39], [57], [64], [68] (with the corresponding Cramér–Rao bounds available in [63]);
   b) specifically in the horizontal orientation in [54].

2) **Two identical dipoles**: a) When both horizontally but orthogonally oriented in spatial collocation, they have been used as a unit for the estimation of the arrival-angles and/or the polarization in [3]–[5], [7], [11]–[13], [15], [18], [21], [26], [28], [32], [45], [51], [54]–[56], [59], [62], [67], [72], and [74].
   b) With one vertical and one horizontal, they are similarly used in [16], [22], [23], [30], [46], [58], and [73].
   c) **Other orthogonally oriented** dipole-pairs are similarly studied in [6], [8], [10], [19], [29], [30], [41], [65], and [79].

Physical implementations of a dipole-pair are presented in [37], [40], [50], [60], [61], [66], and [75].

3) **Two identical loops**: a) When horizontally and orthogonally oriented in spatial collocation, they have been used as a unit for the estimation of the arrival-angles and/or the polarization in [34], [38], [42], [44], and [70].
   b) **Other orthogonally oriented** loop-pairs are similarly studied in [41] and [71].

Hardware implementations and electromagnetics are discussed in [1], [27], [31], [36], [69], and [77].

Closed-form estimation formulas are thus available in the open literature for polarization-parameters (with the incident source’s direction-of-arrival is already known) for three out of fifteen possible antenna/orientation configurations. This work,

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1A dipole is considered “short,” if it is much shorter than a wavelength. A loop is considered “small,” if its circumference is below a quarter-wavelength.

2An orthogonally (not identically) oriented dipole/loop pair has its electromagnetics investigated in [17], [35], [47].
in Section III, will provide closed-form estimation formulas for the other twelve. Section IV will derive and will compare the corresponding Cramér–Rao lower bounds in polarization-estimation for these 15 antenna/orientation configurations. Lastly, Section V will contrast these 15 configurations’ intrinsic numerical stability, with respect to the matrix-inversion implicitly involved in the polarization estimation, given prior knowledge of the direction-of-arrival.

II. ANTENNA-PAIR’S ARRAY MANIFOLD

A fully polarized transverse electromagnetic wave is characterized by its six-component electromagnetic-field vector ([2.5]–(2.8) and (2.11)–(2.12) in [9] with the noise there set to zero):

\[
\begin{bmatrix}
    e_x \\
    e_y \\
    e_z \\
    h_x \\
    h_y \\
    h_z
\end{bmatrix} = \begin{bmatrix}
    \cos \phi \cos \theta & \sin \phi \cos \theta & -\sin \theta \\
    \sin \phi \cos \theta & \cos \phi \cos \theta & \sin \theta \\
    \sin \phi \sin \theta & \cos \phi \sin \theta & \sin \phi \cos \phi \\
    \cos \phi \sin \theta & -\sin \phi \sin \theta & \cos \phi \cos \theta \\
    \cos \phi \cos \theta & \sin \phi \cos \theta & \cos \phi \sin \theta \\
    \sin \phi \cos \theta & -\cos \phi \cos \theta & \sin \phi \sin \theta
\end{bmatrix} \begin{bmatrix}
    v_1 \\
    v_2
\end{bmatrix}
\]

(1)

where \( \theta \in [0, \pi] \) signifies the emitter’s elevation-angle measured from the positive z-axis, \( \phi \in [0, 2\pi] \) denotes the corresponding azimuth-angle measured from the positive x-axis, \( \gamma \in [0, \pi/2] \) refers to the auxiliary polarization angle, \( \eta \in [-\pi, \pi] \) symbolizes the polarization phase difference, \( v_1 = \begin{bmatrix} \cos \phi \cos \theta \sin \phi \cos \theta \sin \theta \end{bmatrix}^T \), \( v_2 = \begin{bmatrix} -\sin \phi \sin \theta \cos \phi \cos \theta \sin \theta \end{bmatrix}^T \), and the superscript \( T \) denotes transposition. Note that \( \Theta \) depends on the incident source’s direction-of-arrival (DOA) and enjoys a block-radialoid structure, whereas \( \Phi \) depends on only the source’s polarization state.

If the receive-antennas are either dipoles and/or loops, and if these receive-antennas orient along some Cartesian axes, then the array manifold would equal the two corresponding components of the six-element vector in (1). Thus, \( \{0\}/(2\lambda) \times 15 \) different antenna/orientation configurations are possible for such an antenna-pair. For this selection of 2 out of 6 elements, it may be represented by a \( 2 \times 6 \) selection-matrix \( S \), which has a “1” on each row, but zeroes elsewhere.

All subsequent analysis will allow these two receive-antennas to be spatially separated or collocated—the latter obviously represents a degenerate case of the former. More mathematically, let one antenna be located at the Cartesian origin, with the other antenna at a known location \( (\Delta_x, \Delta_y, \Delta_z) \), without loss of generality. Then, a spatial phase-factor of \( \exp\{-j'/(\lambda)(\Delta_x \sin \delta \sin \phi + \Delta_y \sin \theta \cos \phi + \Delta_z \cos \theta)\} \) would exist between these two antennas. Each of these 15 configurations would have a \( 2 \times 1 \) array-manifold:

\[
a = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} e^{-j \frac{2\pi}{\lambda} \left( \Delta_x \sin \delta \sin \phi + \Delta_y \sin \theta \cos \phi + \Delta_z \cos \theta \right)} S \begin{bmatrix}
e \h
\end{bmatrix} - \frac{d^2}{D} \Theta
\]

(2)

The question is: which of these fifteen antenna-pair options would be “best,” to estimate \( \gamma \) and \( \eta \) unambiguously. This work will investigate this issue, with regard to the numerical stability, the estimation accuracy, and the estimation validity-region of each antenna/orientation configuration.

Note that a number of relationships exist among the array manifolds of these 15 configurations:

\[
\begin{align*}
\mathbf{a}_{(\Delta_x h_y)}(\hat{\theta}, \hat{\phi}, \gamma, \eta) &= \mathbf{a}_{(\Delta_x h_x)} \left( \hat{\theta}, \hat{\phi} + \frac{\pi}{2}, \gamma, \eta \right) \\
\mathbf{a}_{(\Delta_x h_z)}(\hat{\theta}, \hat{\phi}, \gamma, \eta) &= \mathbf{a}_{(\Delta_x e_z)}(\hat{\theta}, \hat{\phi} + \frac{\pi}{2}, \gamma, \eta) \\
\mathbf{a}_{(\Delta_x e_z)}(\hat{\theta}, \hat{\phi}, \gamma, \eta) &= \mathbf{a}_{(\Delta_x e_z)}(\hat{\theta}, \hat{\phi} - \frac{\pi}{2}, -\gamma, -\eta) e^j\eta \\
\mathbf{a}_{(\Delta_x e_z)}(\hat{\theta}, \hat{\phi}, \gamma, \eta) &= \mathbf{a}_{(\Delta_x e_z)}(\hat{\theta}, \hat{\phi} - \frac{\pi}{2}, -\gamma, -\eta) e^j\eta \\
\mathbf{a}_{(\Delta_y h_z)}(\hat{\theta}, \hat{\phi}, \gamma, \eta) &= \mathbf{a}_{(\Delta_y e_z)} \left( \hat{\theta}, \hat{\phi} + \frac{\pi}{2}, \gamma, \eta \right) \\
\mathbf{a}_{(\Delta_y e_z)}(\hat{\theta}, \hat{\phi}, \gamma, \eta) &= \mathbf{a}_{(\Delta_y e_z)} \left( \hat{\theta}, \hat{\phi} - \frac{\pi}{2}, -\gamma, -\eta \right) e^j\eta
\end{align*}
\]

III. POLARIZATION-ESTIMATION FORMULAS FOR ALL 15 ANTENNA/ORIENTATION CONFIGURATIONS

Eigen-based parameter-estimation algorithms typically involve an intermediate step, that estimates each incident source’s steering vector, correct to within an unknown complex-value scalar \( c \). That is, available (for each incident source)\(^3\) is the estimate \( \hat{\mathbf{a}} \approx \mathbf{c} \mathbf{a} \), from which \( \gamma \) and \( \eta \) are to be estimated. (This approximation becomes equality in noiseless or asymptotic cases.) Hence, there exist two equations and two unknowns. Algebraic manipulation of the two equations yields the estimation formulas of \( \gamma \) and \( \eta \). For example, consider the \( \{e_x, h_x\} \) pair (i.e., configuration 1A of Table I). The two equations are the two rows of

\[
\mathbf{c} \mathbf{a} = c \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} e^{-j \frac{2\pi}{\lambda} \left( \Delta_x \sin \delta \sin \phi + \Delta_y \sin \theta \cos \phi + \Delta_z \cos \theta \right)} \times \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix} \Theta \begin{bmatrix}
\sin \gamma e^{j\eta} \\
\cos \gamma
\end{bmatrix}.
\]

(3)

\(^3\)This does NOT presume only a single source impinging upon the receiver. There could be multiple sources. Moreover, these several sources could possibly be cross-correlated, broadband, and/or time-varying.
TABLE I
POLARIZATION-ESTIMATION FORMULAS OF THE 15 ANTENNA/ORIENTATION CONFIGURATIONS

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Antenna Pair</th>
<th>$\gamma$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A (e_x, h_y)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>1B (e_y, h_x)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>1C (e_z, h_z)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>2A (e_x, e_y)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>2B (h_x, h_y)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>3A (e_x, e_z)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>3B (h_x, h_z)</td>
<td>tan^{-1} $\left( \frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi) \right)$</td>
<td>$\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi)$</td>
<td></td>
</tr>
<tr>
<td>4A (e_y, e_z)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi))$</td>
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<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
<tr>
<td>5A (e_x, e_y)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
<tr>
<td>5B (h_x, h_y)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
<tr>
<td>6A (e_y, e_z)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
<tr>
<td>6B (h_y, h_z)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
<tr>
<td>7A (e_x, e_z)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi - a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi + a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
<tr>
<td>7B (h_x, h_z)</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{cos} \phi + a_2 \text{cos} \theta \text{sin} \phi))$</td>
<td>$\text{atan}^2 (\frac{1}{2} (a_1 \text{cos} \theta \text{sin} \phi - a_2 \text{cos} \theta \text{cos} \phi))$</td>
<td></td>
</tr>
</tbody>
</table>

$\text{atan}$ denotes the angle of the ensuring entity, and $[\cdot]_i$ refers to the $i$th element of the vector inside the square brackets.

The two unknowns are $\gamma$ and $\eta$, with $\Delta_x, \Delta_y, \Delta_z, \theta, \phi, \lambda$ (thus $\Theta$) already known.4

Table I lists the estimation formulas of $\hat{\gamma}$ and $\hat{\eta}$ for each of the 15 antenna/orientation configurations.5 There, $B = (2\pi/\lambda)(\Delta_x \text{sin} \theta \text{sin} \phi + \Delta_y \text{sin} \theta \text{cos} \phi + \Delta_z \text{cos} \theta)$. These estimation-formulas are applicable for the entire validity-region of $\gamma \in [0, (\pi/2)]$ and $\eta \in [-\pi, \pi]$. These estimation-formulas are new to the open literature, to the best knowledge of the present authors, except those few antenna/orientation configurations with references cited in Table I.

It is noteworthy that the vertically oriented “OLD array” (i.e., the $(e_x, h_z)$ configuration) has estimation-formulas independent of the source’s direction-of-arrival; hence, $\hat{\gamma}$ and $\hat{\eta}$ there require no prior information of $\beta$ and $\phi$. This independence applies to no other antenna/orientation configuration. This vertically oriented “OLD array” is unique in this regard: 1) Its array manifold is independent of the azimuth-angle $\phi$. 2) Though its array manifold depends on the elevation-angle $\theta$ through (and only through) $\sin \theta$, this $\sin \theta$ factor is common to both entries in the array manifold. Hence, the ratio between these two entries would be independent of $\theta$.

IV. CRAMÉR–RAO BOUNDS, CRB($\gamma$) and CRB($\eta$), FOR ALL 15 ANTENNA/ORIENTATION CONFIGURATIONS OF SECTION III

To avoid unnecessary distraction from the present objective to compare among the fifteen antenna/orientation configurations, a very simple statistical data model will be used here for the Cramér–Rao bound derivation: the received signal $s(t) = e^{j(2\pi f_0 + \theta t)}$ is a pure tone at unity-power, a known frequency of $f_0$, and a known initial phase of $\theta$. At the $m$th time-instance of $t = mT_s$, the antenna-pair collects the $2 \times 1$ data-vector,

$$\mathbf{z}(mT_s) = \mathbf{a}_s(mT_s) + \mathbf{n}(mT_s)$$

where $T_s$ refers to the time-sampling period, $\mathbf{n}(t)$ denotes a $2 \times 1$ vector of spatio-temporally uncorrelated zero-mean complex-value Gaussian additive noise, with an unknown deterministic covariance-matrix of $\mathbf{G}_0 = \text{diag}(\sigma^2, \sigma^2)$, where $\sigma^2$ representing the known noise-variance at each antenna.

With $M$ number of time-samples, the $2M \times 1$ collected data-set equals

$$\zeta = [\mathbf{z}(T_s)]^T, \ldots, [\mathbf{z}(MT_s)]^T$$

where $\mathbf{s} = e^{jT_s\omega}, e^{j2T_s\omega}, \ldots, e^{jMT_s\omega}$, $\otimes$ symbolizes the Kronecker product, $\mathbf{v}$ represents a $2M \times 1$ noise vector with a spatio-temporal covariance matrix of $\mathbf{G} = I_M \otimes \mathbf{G}_0$, and $I_M$ denotes an $M \times M$ identity matrix. Therefore, $\zeta \sim N(\mu, \mathbf{G})$, i.e., a complex-value Gaussian vector with mean $\mu$ and covariance $\mathbf{G}$.

The to-be-estimated $\gamma$ and $\eta$ are modeled as deterministic. Collect all deterministic unknown entities into the $2 \times 1$ vector

$$\mathbf{u}$$
of $\psi = [\gamma, \eta]^T$. The resulting $2 \times 2$ Fisher information matrix (FIM)
\[
J = \begin{bmatrix}
J_{\gamma,\gamma} & J_{\gamma,\eta} \\
J_{\eta,\gamma} & J_{\eta,\eta}
\end{bmatrix}
\]
has its $(i, j)$th entry equal to (see [25, (8.34)]):
\[
[J]_{i,j} = 2\text{Re} \left[ \left( \frac{\partial \psi}{\partial \psi} \right)^H \Gamma^{-1} \left( \frac{\partial \psi}{\partial \psi} \right) \right] + \text{Tr} \left[ \Gamma^{-1} \frac{\partial \Gamma}{\partial \psi} \Gamma^{-1} \frac{\partial \Gamma}{\partial \psi} \right]
\]
where $\text{Re}[\cdot]$ denotes the real-value part of the entity inside $[\cdot]$, $\text{Tr}[\cdot]$ represents the trace operator, and
\[
\frac{\partial \psi}{\partial \gamma} = \frac{\partial a}{\partial \gamma} \otimes s
\]
\[
\frac{\partial \psi}{\partial \eta} = \frac{\partial a}{\partial \eta} \otimes s
\]
\[
\frac{\partial a}{\partial \gamma} = \Omega \begin{bmatrix}
\cos(\gamma) e^{i\eta} \\
-\sin \gamma
\end{bmatrix}
\]
\[
\frac{\partial a}{\partial \eta} = \Omega \begin{bmatrix}
j \sin(\gamma) e^{i\eta} \\
0
\end{bmatrix}
\]
where $\Omega \triangleq D \Theta$. The elements of the FIM equal
\[
J_{\gamma,\gamma} = \frac{2M}{\sigma^2} \left[ \left( \begin{bmatrix} \Theta_{1,1} \end{bmatrix}^2 + \left( \begin{bmatrix} \Theta_{2,1} \end{bmatrix}^2 \right) \right) \cos(\gamma)^2 + \left( \begin{bmatrix} \Theta_{2,2} \end{bmatrix}^2 \right) \sin(\gamma)^2 \right] - \sin(2\gamma) \cos(\eta) \]
\[
J_{\gamma,\eta} = \frac{2M}{\sigma^2} \left[ c_1(\cos(\gamma)^2) + c_2(\sin(\gamma)^2) \right] - c_3 \sin(2\gamma) \cos(\eta)
\]
\[
J_{\eta,\gamma} = \frac{2M}{\sigma^2} \left[ \left( \begin{bmatrix} \Theta_{2,1} \end{bmatrix}^2 + \left( \begin{bmatrix} \Theta_{2,2} \end{bmatrix}^2 \right) \right) \sin(\gamma)^2 \right] - \sin(2\gamma) \cos(\eta)
\]
\[
J_{\eta,\eta} = \frac{2M}{\sigma^2} \left[ c_1(\cos(\gamma)^2) + c_2(\sin(\gamma)^2) \right] - c_3 \sin(2\gamma) \cos(\eta)
\]
where $c_1 = \left( \begin{bmatrix} \Theta_{1,1} \end{bmatrix}^2 + \left( \begin{bmatrix} \Theta_{2,2} \end{bmatrix}^2 \right) \right)$, $c_2 = \left( \begin{bmatrix} \Theta_{2,1} \end{bmatrix}^2 + \left( \begin{bmatrix} \Theta_{2,2} \end{bmatrix}^2 \right) \right)$, and $c_3 = \left( \begin{bmatrix} \Theta_{1,1} \end{bmatrix}^2 + \left( \begin{bmatrix} \Theta_{2,1} \end{bmatrix}^2 + \left( \begin{bmatrix} \Theta_{2,2} \end{bmatrix}^2 \right) \right)$, with $i,j$ indexing the $(i,j)$th entry of the matrix in $[\cdot]$.

The polarization-estimation Cramér–Rao bounds equal
\[
\text{CRB}(\gamma) = [J^{-1}]_{1,1} = \frac{J_{\gamma,\gamma}}{J_{\gamma,\gamma} J_{\eta,\eta} - (J_{\gamma,\eta})^2}
\]
\[
\text{CRB}(\eta) = \frac{J_{\eta,\eta}}{J_{\gamma,\gamma} J_{\eta,\eta} - (J_{\gamma,\eta})^2}
\]
\[
\text{CRB}(\gamma) = \left[ J^{-1} \right]_{1,1} = \frac{\sigma^2}{2M} \left[ \frac{c_1(\cos(\gamma)^2) + c_2(\sin(\gamma)^2) - c_3 \sin(2\gamma) \cos(\eta)}{c_1^2(\cos(\gamma)^2) + c_2^2(\sin(\gamma)^2) - c_3^2 \sin(2\gamma) \cos(\eta)} \right]
\]

Tables II lists the closed-form formulas for $\text{CRB}(\gamma)$ and $\text{CRB}(\eta)$, explicitly in terms of the data-model parameters. These Cramér–Rao bounds would be unchanged by any prior known spatial separation between the two receive-antennas, and these Cramér–Rao bounds are independent also of the value of the frequency, if prior known. This is because the values of the spatial separation and of the frequency would affect only $B$ in $D$, which is canceled in the course of calculating the FIM.

Fig. 1(a)-(c) plots $(2M)/(\sigma^2)\text{CRB}(\gamma) = (2M)/(\sigma^2) \text{CRB}(\eta)$ for the three “OLD arrays” of configurations 1A-1C of Table I. Fig. 2(a) and (b) plots $\text{CRB}(\gamma)$ and $\text{CRB}(\eta)$ of the horizontally oriented dipole-pair (i.e., configuration 2A). Fig. 2(c) and (d) does the same for the horizontally loop-pair (i.e., configuration 2B). Each remaining configuration has $\text{CRB}(\gamma)$ and $\text{CRB}(\eta)$ dependent on three or more independent variables, thus not fully representable by any three-dimensional graph.

Below are some qualitative observations on these derived Cramér–Rao bounds.

1) The horizontally oriented “OLD arrays” ($\{e_x, h_z\}$ of configuration 1A, and $\{e_y, h_y\}$ of configuration 1B) offer finite Cramér–Rao bounds for sources impinging not horizontally. Please see Figs. 1(a) and (b). In contrast, the vertically oriented “OLD array” ($\{e_z, h_y\}$ of configuration 1C) offers finite Cramér–Rao bounds for sources impinging horizontally. Please see Fig. 1(c).

2) The horizontally oriented “OLD arrays” ($\{e_x, h_z\}$ of configuration 1A, and $\{e_y, h_y\}$ of configuration 1B) has a $\text{CRB}(\gamma)$ independent of the polarization parameters of $\gamma$ and $\eta$. The vertically oriented “OLD array” ($\{e_z, h_y\}$ of configuration 1C) has both $\text{CRB}(\gamma)$ and $\text{CRB}(\eta)$ independent of the azimuth-angle $\phi$.

3) The Cramér–Rao bounds of the $\{e_x, h_z\}$ “OLD array” are related to the Cramér–Rao bounds of the other horizontally oriented “OLD array” of $\{e_y, h_y\}$, by substituting $\phi$ by $\phi + (\pi/2)$. This is because $a_{1,2}(\theta, \phi, \gamma, \eta) = a_{1,1}(\theta, \phi, (\pi/2), \gamma, \eta)$. Similar relationships exist between configurations 3A and 4A, between configurations 3B and 4B, between configurations 6A and 7A, and between configurations 6B and 7B.

4) For configurations 3A-7B, both $\text{CRB}(\gamma)$ and $\text{CRB}(\eta)$ depend on the direction-of-arrival (i.e., $\theta$ and $\phi$) and the polarization (i.e., $\gamma$ and $\eta$). Other dependencies are listed in Table III.

5) $\text{CRB}(\eta) \to \infty$ as $\gamma \to 0$ for all 15 configurations.
TABLE II
CRAMÉR–RAO BOUNDS FOR POLARIZATION-ESTIMATES FOR THE 15 ANTENNA/ORIENTATION CONFIGURATIONS

<table>
<thead>
<tr>
<th>Antenna/Orientation</th>
<th>Cramer–Rao Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A (r_x, r_x)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>1B (r_y, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>1C (r_x, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>2A (r_x, r_z)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>2B (r_y, r_z)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>2C (r_x, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>3A (r_x, r_x)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>3B (r_y, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>3C (r_x, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>4A (r_x, r_x)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>4B (r_y, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>5A (r_x, r_x)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>5B (r_y, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>6A (r_x, r_x)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>6B (r_y, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>7A (r_x, r_x)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
<tr>
<td>7B (r_y, r_y)</td>
<td>( \frac{1}{\sigma_1^2} )</td>
</tr>
</tbody>
</table>

In this table, \( \sigma_1^2 \) is the variance of the estimated parameter.

V. CONDITION NUMBERS ASSOCIATED WITH THE ESTIMATION-FORMULAS OF SECTION III

The estimate \( \hat{g} \) could be construed as

\[
\hat{g} = \left( \frac{\mathbf{D} \mathbf{g}}{\mathbf{S} \mathbf{g}} \right)^{-1} = \mathbf{S}^{-1} \mathbf{D} \mathbf{g}
\]

where \( \mathbf{g}(t) \) denotes the 2x1 data collected at the time-instant \( t \).

The above requires an inversion of the 2x2 matrix \( \mathbf{D} = \mathbf{S} \mathbf{g} \). The numerical robustness in computing \( \hat{g} \) thus depends on the condition number \( \kappa_2(\mathbf{S}) \approx \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \) of \( \mathbf{S} \), where \( \lambda_{\text{max}} \) denotes the largest-magnitude eigenvalue of the matrix \( \mathbf{S} \).

If \( \mathbf{g} \) were estimated without the above matrix inversion, this condition-number analysis would be irrelevant.
symbolizes the smallest-magnitude eigenvalue. If \( \Omega \) approaches singularity, \( K_2 \) would approach infinity. The numerically most stable matrix-inversion corresponds to \( K_2 = 1 \).

Table IV lists the condition number for each of the 15 different composition/orientation configurations, derived by the present authors. Because \( K_2 \) depends only on \( \Omega \), \( K_2 \) is independent of \( \gamma \) and \( \eta \), but depends only on \( \theta \) and \( \phi \).

The ideal \( K_2 = 1 \) is attained by only the three “OLD array” compositions/orientations of \( \{ e_x, h_x \}, \{ e_y, h_y \}, \) and \( \{ e_z, h_z \} \). For the other 12 configurations (i.e., configurations 2A-2B to 7A-7B), Fig. 3(a)–(f) plots their condition numbers versus \( \theta \in [0, \pi] \) and versus \( \phi \in [0, 2\pi] \). Each figure shows the values of \( \theta \) and \( \phi \) where \( K_2 \to \infty \). These are summarized in the two right-most columns of Table IV.

Below are some qualitative observations:

6) The condition numbers are all independent of the spatial separation between the two component-antennas, and independent of the frequency. This is because the matrix \( D \) always has a unity condition number, regardless of the aforementioned entities. Hence, \( K_2 \) is unaffected by these entities.

7) For and only for the three “OLD array” configurations (i.e., a dipole and a loop in an identical orientation), \( K_2 = 1, \forall \theta, \phi \). The reason is as follows: From (1)

\[
\mathbf{D} \mathbf{\Theta} = \mathbf{D} \begin{bmatrix} \mathbf{v}_1 \end{bmatrix}_m \begin{bmatrix} \mathbf{v}_2 \end{bmatrix}_m = \Omega
\]

with \( \begin{bmatrix} \cdot \end{bmatrix}_m \) referring to the \( m \)th entry of the vector inside the square brackets. The radialoid structure of \( \Omega \) gives
\[ \Omega^H \Omega = (|v_1|^2 + |v_2|^2)I_2, \]
where \( I_2 \) symbolizes a \( 2 \times 2 \) identity matrix. Hence, the condition number equals 1.\(^7\)

8) For all the dipole-pair configurations and all the loop-pair configurations (i.e., configurations 2A-2B, 3A-3B, 4A-4B), each “A” configuration (such as 2A) can lead to the corresponding “B” configuration (such as 2B), by switching \( e \) with \( h \). More precisely, with \( \Omega = \Omega_A \) for the “A” configuration and with \( \Omega = \Omega_B \) for the “B” configuration, then \( \Omega_B = \Omega_A P \), where \( P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) denotes a 90°-rotation matrix. That is, \( \Omega_A \) and \( \Omega_B \) are similar matrices. Hence, configuration 2A must have the same condition number as configuration 2B. Similar reasoning applies between 3A and 3B, and between 4A and 4B.

9) For non-identically oriented dipole/loop antenna-pairs (i.e., configurations 5A-5B to 7A-7B), each “A’” configuration (such as 5A) can lead to the corresponding “B” configuration (such as 5B), also by switching \( e \) with \( h \). Consider configurations 5A and 5B, \( \Omega_B = \mathbf{R} \Omega_A \mathbf{P} \), where \( \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) denotes a reflection matrix. Hence, \( \Omega_A \) and \( \Omega_B \) have the same condition number. Consequently, configuration 5A must have the same condition number as configuration 5B. Similar reasoning applies between 6A and 6B, and between 7A and 7B.

10) The condition number of configurations 3A-3B, with a \((\pi)/2\) shift in \( \phi \), leads to the condition number of configurations 4A-4B. This is because the \( x \)-axis in 3A-3B corresponds to the \( y \)-axis in 4A-4B. Similarly, this holds between configurations 6A-6B on one hand and configurations 7A-7B on the other hand.

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\( \gamma \) (degrees) \( \theta \) (degrees)

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Fig. 2. (a) \( (2M)/(\sigma^2)\)CRB(\( \gamma \)) plotted versus \( \theta \) and \( \gamma \), for configuration 2A (i.e., \{\( e_x, e_y \}\}). (b) \( (2M)/(\sigma^2)\)CRB(\( \eta \)) plotted versus \( \theta \) and \( \gamma \), for configuration 2A (i.e., \{\( e_x, e_y \}\}). (c) \( (2M)/(\sigma^2)\)CRB(\( \gamma \)) plotted versus \( \theta \) and \( \gamma \), for configuration 2B (i.e., \{\( h_x, h_y \}\}). (d) \( (2M)/(\sigma^2)\)CRB(\( \eta \)) plotted versus \( \theta \) and \( \gamma \), for configuration 2B (i.e., \{\( h_x, h_y \}\}).
For each antenna/orientation configuration, the condition number is symmetric with respect to $\theta = (\pi)/2$ along the $\theta$-axis, and with respect to $\phi = \pi$ along the $\phi$-axis.

VI. CONCLUSION

Investigated herein are all 15 possible antenna/orientation configurations that measure any 2 of the 6 components of the electromagnetic-field vector. Obtained for each antenna/orientation configuration are the closed-form polarization-estimation formulas, the corresponding validity region for unambiguous estimation, the corresponding Cramér–Rao bounds, and the corresponding condition number as a function of the azimuth-elevation direction-of-arrival. The same derived results would apply, whether each pair is collocated or spatially separated. Among these 15 configurations, especially advantageous is the vertically oriented “OLD array” (i.e., the configuration), which requires no prior information of the source’s direction-of-arrival for polarization, but offers an ideal condition number of unity, and has finite Cramér–Rao bounds except for near-vertically incident sources. For the horizontally oriented “OLD arrays” (i.e., the configuration or the configuration), they also offer the ideal condition number of unity, and have finite Cramér–Rao bounds except for near-horizontally incident sources.
Fig. 3. (a) Condition number of $\mathbf{D}$ for configuration 2A (i.e., $\{r_z, r_y\}$) and the configuration 2B (i.e., $\{h_x, h_y\}$). (b) The condition number of $\mathbf{D}$ for configuration 3A (i.e., $\{r_z, r_y\}$) and the configuration 3B (i.e., $\{h_x, h_z\}$). (c) The condition number of $\mathbf{D}$ for configuration 4A (i.e., $\{r_y, r_z\}$) and the configuration 4B (i.e., $\{h_y, h_z\}$). (d) The condition number of $\mathbf{D}$ for configuration 5A (i.e., $\{r_x, h_y\}$) and the configuration 5B (i.e., $\{h_x, r_y\}$). (e) The condition number of $\mathbf{D}$ for configuration 6A (i.e., $\{r_z, h_y\}$) and the configuration 6B (i.e., $\{h_z, r_y\}$). (f) The condition number of $\mathbf{D}$ for configuration 7A (i.e., $\{r_z, h_y\}$) and the configuration 7B (i.e., $\{r_x, h_y\}$).
ACKNOWLEDGMENT

The authors would like to thank Prof. L. Yeh for valuable discussions.

REFERENCES
