Polarisation-sensitive geometric modelling of the distribution of direction-of-arrival for uplink multipaths

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Abstract: This work extends the ‘geometric modelling’ of channel fading to account for the effects of the polarisation states of the transmitting/receiving antennas, and for the distribution of the direction of arrival of the uplink multipaths. The derived formulas are closed-form functions, explicitly in terms of a few model parameters idealising the spatial geometry among the transmitter, the scatterers and the receiver. These formulas fit the empirical data well.

1 Introduction

For multipaths arriving at a receiver, their azimuth angle-of-arrival distribution determines the angular spread and the spatial decorrelation across the spatial aperture of receiving antenna.

‘Geometric modelling’ idealises the spatial geometry among the transmitter, the scatterers and the receiver, such that a few model parameters can encapsulate key aspects of the arriving multipaths’ direction-of-arrival (DOA) distribution (or other fading metrics) for a broad range of field scenarios. This idealised geometry is necessarily site unspecific, but can be widely applicable to a generic category of propagation channels, such as an entire subclass of bad urban situations or the whole subclass of rural scenarios, thereby simplifying the system design of mass-marketable products for a large subclass of propagation channels.

There exists a rich literature on rigorous mathematical derivations of the arriving multipaths’ azimuth-DOA distribution, based on ‘geometric modelling’ of the propagation channel. Examples include:

1. A uniform density within a circular-disc support region of radius \(R\), which is less than the transmitter–receiver separation \(D\) [1–5].

2. A uniform density within a support region of a 2\(\beta\) pie-shaped cut of a circular disc of radius \(R > D\) [6] (for a directional transmitter with a 2\(\beta\) azimuth beam width), or \(R < D\) [7].

3. A uniform density within a hollow circular-disc support region with an outer radius of \(R \leq D\) [8], or \(R > D\) [1, 3, 6].

4. An inverted parabolic density within a circular-disc support region of radius \(R \leq D\) [9].

5. A conical density within a circular-disc support region of radius \(R \leq D\) [1], or \(R > D\) [1].

6. A uniform density within an elliptical-disc support region that is centred at the transmitter but excluding the receiver [2], or that is focused at the transmitter and the receiver [4, 10].


8. A Gaussian density centred at the transmitter [12–14].

Radiowave wireless transmission and reception, however, is intrinsically electromagnetic in nature. Hence, the arriving multipaths would have a DOA distribution that depends on the polarisation states of the transmitting antennas, the receiving antennas and the polarisation properties of the
scatterers. Such polarisation-related considerations are overlooked in all above references. This paper will show how to extend the above customary ‘geometric modelling’ of the uplink multipaths’ DOA distribution, to account for the polarisation states of the transmitting/receiving antennas.

This proposed polarisation–sensitive modelling approach could also be applied to azimuth-elevation DOA modelling, to joint DOA–TOA (time-of-arrival) distribution, to time-varying Doppler-spread behaviours and so on.

This polarisation-sensitive extension is herein demonstrated using a Gaussian-distributed cluster. However, this approach is equally valid for other cluster distributions, such as those in 1–8 above. The Gaussian-distributed cluster is chosen here, because it offers robust and reliable fitting of the open literature’s empirical data, according to an exhaustive comparison [15] across various geometric models for the azimuth-angle distribution.

2 Proposed statistical model

A mobile, located at $z_{\text{MS}} = [x_{\text{MS}}, y_{\text{MS}}]$ on a two-dimensional Cartesian plane $\mathbb{R}^2$, emits a fully polarised signal. This emitted signal bounces off each scatterer before reaching the base station at $z_{\text{BS}} = (0, 0)$. Each scatterer acts as a polarisation-sensitive re-transmitter, producing a fully polarised multipath towards each receiving antenna at the base station. That is, if $S$ number of scatterers exist in $\mathbb{R}^2$, then $S$ multipaths will travel from the transmitter to each receiving antenna, with each such multipath representing one ‘bounce’ off a different scatterer out of the $S$ scatterers. The polarisation and the power of this reflected ray depend on the polarisation of the incident ray and on the scatterer’s intrinsic properties (to be defined below).

2.1 Scatterer as a polarisation-sensitive re-transmitter

From basic electromagnetics, any fully polarised electromagnetic wave may be decomposed as the sum of a vertically polarised wave and a horizontally polarised wave. When this fully polarised electromagnetic ray reflects (scatters) off a surface (a scatterer), the reflected (scattered) ray’s polarisation state will depend on the incident wave’s polarisation state, frequency, incident angle and the reflector’s (scatterer’s) electromagnetic properties and surface roughness.

More mathematically, let a unit-power incoming ray have vertically polarised power $a_v^2$ and horizontally polarised power $a_h^2 = 1 - a_v^2$; let the reflected ray have vertically polarised power $b_v$ and horizontally polarised power $b_h$. The relationship among these is idealised as governed by only the $2 \times 2$ scattering matrix $\mathbf{S}$

$$
\begin{bmatrix}
    b_v \\
    b_h
\end{bmatrix}
= \mathbf{S}
\begin{bmatrix}
    a_v \\
    a_h
\end{bmatrix}
$$

(1)

By definition, $a_v^2 + a_h^2 = 1$, $a_v^2 = 1 - a_h^2 = 1$, and all elements in $\mathbf{S}$ are non-negative real numbers not exceeding 1.

All scatterers are herein modelled as belonging to exactly one of the following four types:

Type $v, v$: In this cluster, any scatterer will respond only to a vertically polarised incident ray. The re-transmitted ray will be vertically polarised with amplitude $b_{v,v}$, where

$$
\begin{bmatrix}
    b_{v,v} \\
    0
\end{bmatrix}
= \begin{bmatrix}
    s_{v,v} \\
    0
\end{bmatrix}
\begin{bmatrix}
    a_v \\
    \sqrt{1 - a_v^2}
\end{bmatrix}
$$

(2)

A horizontally polarised incident ray will produce no reflection and no scattering.

Type $b, v$: In this cluster, any scatterer will respond only to a vertically polarised incident ray. The re-transmitted ray will be horizontally polarised with amplitude $b_{b,v}$, where

$$
\begin{bmatrix}
    0 \\
    b_{b,v}
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    s_{b,v}
\end{bmatrix}
\begin{bmatrix}
    a_v \\
    \sqrt{1 - a_v^2}
\end{bmatrix}
$$

(3)

A horizontally polarised incident ray will produce no reflection and no scattering.

Type $v, h$: In this cluster, any scatterer will respond only to a horizontally polarised incident ray. The re-transmitted ray will be vertically polarised with amplitude $b_{v,h}$, where

$$
\begin{bmatrix}
    b_{v,h} \\
    0
\end{bmatrix}
= \begin{bmatrix}
    s_{v,h} \\
    0
\end{bmatrix}
\begin{bmatrix}
    a_v \\
    \sqrt{1 - a_v^2}
\end{bmatrix}
$$

(4)

A horizontally polarised incident ray will produce no reflection and no scattering.

Type $b, h$: In this cluster, any scatterer will respond only to a horizontally polarised incident ray. The re-transmitted ray will be horizontally polarised with amplitude $b_{b,h}$, where

$$
\begin{bmatrix}
    0 \\
    b_{b,h}
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    s_{b,h}
\end{bmatrix}
\begin{bmatrix}
    a_v \\
    \sqrt{1 - a_v^2}
\end{bmatrix}
$$

(5)

A vertically polarised incident ray will produce no reflection and no scattering.

Combining all the above contributions, (1) becomes

$$
b_v = s_{v,v}a_v + s_{v,h}a_h\sqrt{1 - a_v^2}
$$

(6)

$$
b_h = s_{h,v}a_v + s_{h,h}a_h\sqrt{1 - a_v^2}
$$

(7)
2.2 Scatterers’ spatial distribution

All scatterers are modelled as belonging to exactly one of these four clusters, TYPE = v, v, h; v, h; or h, h. These four clusters overlap spatially, co-centring on the transmitter at $z_{MS}$ but possibly having different spatial extents. In each cluster, the scatterers’ spatial density $\Lambda_{\text{TYPE}}(\mathbf{x})$ is circularly symmetric bivariate Gaussian (Fig. 1), with a variance of $\Sigma^2_{\text{TYPE}}$, and centred at $z_{MS}$. That is

$$\Lambda_{\text{TYPE}}(\mathbf{x}) = \frac{1}{2\pi \Sigma^2_{\text{TYPE}}} \exp\left(-\frac{|z - z_{MS}|^2}{2\Sigma^2_{\text{TYPE}}}\right)$$  \hspace{1cm} (8)

The above expression generalises the uni-Gaussian spatial distribution (8 in Section 1) in [12–14, 16], wherein all scatterers belong to the same non-polarised cluster of exactly the same spatial distribution as in (8) but with only one possible value to ‘TYPE’. This polarised tetra-cluster generalisation could apply to 1–7 in Section 1, by substituting the cluster distributions there for (8).

2.3 Listing of the model parameters

The model has six degrees-of-freedom altogether:

i. $\Sigma_{v,v}/D$, $\Sigma_{h,h}/D$, $\Sigma_{v,h}/D$, and $\Sigma_{h,v}/D$ refer to the normalised scatterer spatial spreads of the four kinds of polarisation-sensitive clusters. The normalisation is with respect to the distance $D$ between the transmitter and the receiver.

ii. The two parameters $s_{v,v}$ and $s_{h,h}$ determine the entire $2 \times 2$ scattering matrix $S$.

The value of $a_s$ is determined by the transmit antenna’s actual polarisation state.

The tetra-Gaussian polarisation-sensitive cluster modelling here differs from the bi-cluster in [17] (which models for the spatial correlation coefficient across a receiver’s aperture, that is, the correlation coefficient between measurements by two receiving antennas displaced in space). The latter could be construed as a degenerate case of the former, with $\Sigma_{h,v} = \Sigma_{v,v}$ and $\Sigma_{v,h} = \Sigma_{h,h}$. The present model needs $\Sigma_{h,v} \neq \Sigma_{v,v}$ and $\Sigma_{v,h} \neq \Sigma_{h,h}$, such that a different receiving antenna polarisation (for the same transmitting antenna polarisation) could give different DOA distributions.

3 Polarisation-sensitive DOA distribution formula

3.1 Preliminary geometric analysis

For a cluster spatially distributed according to a two-dimensional circular Gaussian density with a standard deviation $\sigma$, its corresponding DOA distribution has been derived [12–14]

$$\frac{1}{2\pi} e^{(\cos \phi - 1)/2\sigma^2} \text{erfc}\left(\frac{-\cos \phi}{\sqrt{2}\sigma}\right),$$

$$\forall \phi \in [0, \pi] \hspace{1cm} (9)$$

The above result applies for each of the four clusters in Section 2.2. For example, the DOA distribution for $h,v$ is obtained from (9) by setting $\sigma = \Sigma_{v,v}/D$, to give

$$f_{v,v}(\phi) = \frac{1}{2\pi} e^{(\cos \phi - 1)/2\sigma^2} \text{erfc}\left(\frac{-\cos \phi}{\sqrt{2}\sigma}\right),$$

$$\forall \phi \in [0, \pi] \hspace{1cm} (10)$$

Similar expressions hold for $f_{h,v}(\phi), f_{v,h}(\phi)$ and $f_{h,h}(\phi)$.

For the distribution $f_c(\phi)$ of the $h,v$ over $\phi$, it is a weighted average of $f_{v,v}(\phi)$ (vertically polarised transmission that maintains its vertical polarisation despite reflection and scattering) and $f_{v,h}(\phi)$ (horizontally polarised transmission that cross-scattered to become vertically polarised). More precisely

$$f_c(\phi) = \frac{s_{v,v}a_s}{\int_0^{2\pi} s_{v,v}a_s f_{v,v}(\phi) + s_{v,h} \sqrt{1 - a_s^2} f_{v,h}(\phi) \, d\phi} \times \frac{1}{\int_0^{2\pi} s_{v,v}a_s f_{v,v}(\phi) + s_{v,h} \sqrt{1 - a_s^2} f_{v,h}(\phi) \, d\phi}$$

$$= \frac{s_{v,v}a_s f_{v,v}(\phi) + s_{v,h} \sqrt{1 - a_s^2} f_{v,h}(\phi)}{s_{v,v}a_s + s_{v,h} \sqrt{1 - a_s^2}}$$ \hspace{1cm} \text{where the denominator normalises $f_c(\phi)$ to have unity area} \hspace{1cm} (11)$$

The tetra-Gaussian polarisation-sensitive cluster modelling here differs from the bi-cluster in [17] (which models for the spatial correlation coefficient across a receiver’s aperture, that is, the correlation coefficient between measurements by two receiving antennas displaced in space). The latter could be construed as a degenerate case of the former, with $\Sigma_{h,v} = \Sigma_{v,v}$ and $\Sigma_{v,h} = \Sigma_{h,h}$. The present model needs $\Sigma_{h,v} \neq \Sigma_{v,v}$ and $\Sigma_{v,h} \neq \Sigma_{h,h}$, such that a different receiving antenna polarisation (for the same transmitting antenna polarisation) could give different DOA distributions.

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under it. Similarly, for $\sigma_h$

$$f_h(\phi) = \frac{s_{h,v}a_vf_{h,v}(\phi) + s_{h,h}\sqrt{1 - a_v^2}f_{h,h}(\phi)}{\int_0^{2\pi} s_{h,v}a_vf_{h,v}(\phi) + s_{h,h}\sqrt{1 - a_v^2}f_{h,h}(\phi)}d\phi$$

$$= \frac{\sqrt{1 - a_v^2}a_vf_{h,v}(\phi) + \sqrt{1 - a_h^2}\sqrt{1 - a_v^2}f_{h,h}(\phi)}{\sqrt{1 - a_v^2}a_v + \sqrt{1 - a_h^2}\sqrt{1 - a_v^2}}$$

(12)

### 3.2 Qualitative trends in the derived formula

1. $f_v(\phi)$ in Figs. 2a–d: Recall that $f_v(\phi)$ is weighted by the two DOA distributions of $\Sigma_{v,v}/D$ (for vertically polarised transmission that retains its vertical polarisation while passing through the $(v, v)$ cluster) and $\Sigma_{v,h}/D$ (for the transmitter’s horizontally polarised component that crosses into vertical polarisation at the receiver, because of scattering through the $(v, h)$ cluster). Figs. 2a–c demonstrate how $f_v(\phi)$ of (11) is affected by the relative weights of the $(v, v)$ cluster and the $(v, h)$ cluster. Figs. 2a–c are plotted at $(s_{v,v}a_v)/(s_{v,h}\sqrt{1 - a_v^2}) = 1$; hence, the emitted power is evenly distributed between the vertically and horizontally polarised components, at transmission.

Fig. 2d plots the DOA distribution for the customary uni-Gaussian distributed cluster, to help in analysing Figs. 2a–c later. For a spatially concentrated cluster (i.e. small $\sigma = \Sigma/D$), the multipaths arrive mostly along the line-of-sight direction of $\phi = 0$. For $\phi$ farther away from zero, fewer multipaths arrive. However, as the cluster’s spatial spread $\sigma = \Sigma/D$ increases, the multipaths arrive over a wider range of azimuth angles and the line-of-sight azimuth direction gradually loses its dominance.

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**Figure 2** DOA distribution $f_v(\phi)$ in (11) of the vertically polarised component at the receiver, against $\phi$ and $\Sigma_{v,v}/D$

- $a$ $s_{v,v}a_v/s_{v,h}a_h = s_{v,v}a_v/s_{v,h}\sqrt{1 - a_v^2} = 1$ and $\Sigma_{v,h}/D = 10$
- $b$ $s_{v,v}a_v/s_{v,h}a_h = s_{v,v}a_v/s_{v,h}\sqrt{1 - a_v^2} = 1$ and $\Sigma_{v,h}/D = 1$
- $c$ $s_{v,v}a_v/s_{v,h}a_h = s_{v,v}a_v/s_{v,h}\sqrt{1 - a_v^2} = 1$ and $\Sigma_{v,h}/D = 0.1$
- $d$ DOA distribution in (9) for a uni-Gaussian-distributed cluster. This figure is to aid the interpretation of Figs. 2a and c
Figure 3  DOA distribution $f_v(\phi)$ in (11) of the vertically polarised component at the receiver, against $\phi$ and $(s_v v_{a_v})/(s_h v_{a_h}) = (s_v v_{a_v})/(s_h v_{a_h} \sqrt{1 - a_v^2})$.

Here, $\Sigma_{a_v}/D = 5$, $\Sigma_{a_h}/D = 1$, and $s_{a_v} \sqrt{1 - a_v^2} = 0.5$.

The peak in Fig. 2a is mainly because of the $(v, v)$ cluster, as the $(v, h)$ cluster there (at a large $(\Sigma_{a_v}/D = 10)$ would have a broad DOA spread. As the $(v, h)$ cluster obtains more spatially concentrated from Fig. 2a to Fig. 2c, the $(v, h)$ cluster will narrow its DOA spread, thereby enhancing the peak along the entire $(\Sigma_{a_v}/D)$ axis.

2. $f_v(\phi)$ in Fig. 3: It investigates how $f_v(\phi)$ is affected by the relative weight of $s_{a_v} a_v$ and $s_{a_h} a_h \sqrt{1 - a_v^2}$. There, $(\Sigma_{a_v}/D = 1 < (\Sigma_{a_h}/D = 5)$, such that the $(v, h)$ cluster is more likely than the $(v, v)$ cluster to contribute towards a peak around the line-of-sight azimuth DOA, with all else being equal about these two clusters. As $(s_{a_v} a_v)/(s_{a_h} a_h)$ decreases, the horizontally polarised transmission becomes more important than the vertically polarised transmission, thereby amplifying the effects of the $(v, h)$ cluster over the $(v, v)$ cluster to produce a prominent line-of-sight peak.

3. On $f_h(\phi)$: As $f_h(\phi)$ has the same mathematical form as $f_v(\phi)$, the above discussion is equally applicable to $f_h(\phi)$ with the subscripts $v$ and $h$ interchanged in all mathematical terms.

4 Validation of derived azimuth-DOA distribution by empirical data

The above derived polarisation-sensitive azimuth-angle distributions in (11) and (12) are validated by empirical data from the open literature. (The authors’ exhaustive literature search identifies only one such data set.)

1. Most such empirical measurement papers do not describe the polarisation state or orientation of the transmit antennas and/or receive antennas. If they do, the description is often vague, like ‘elliptical polarisation’ (without specifying the exact polarisation parameter values) [18] or ‘dual polarised’ (without specifying the two polarisations) [19–23].

2. For those papers that adequately describe the polarisation, the electromagnetic signal was often transmitted and/or measured the using only one polarisation. Very few papers present empirical data for different polarisations, while maintaining the same transmitter–receiver locations and in the same field environment [24–34].

3. For the very few papers that used dual polarisation at the transmitter and/or the receiver, and that sufficiently specify the polarisation states: the data sets are sometimes numerically illegible, in that no numerical value can be reliably extracted from the graphs which are often contour maps or are three-dimensional maps of closely clustered spikes [21–23, 26–29, 31–34].

Formulas in (11) and (12) are jointly calibrated against the empirical data in [35, Fig. 10], by minimising the curve-fitting mean-square error (MSE)

$$\text{MSE} \left( \frac{\Sigma_{s_v}}{D}, \frac{\Sigma_{s_h}}{D}, \frac{\Sigma_{s_v}}{D}, \phi_{LOS} \right) = \frac{1}{N_v + N_h} \left\{ \sum_{i=1}^{N_v} [y_v(i) - f_v(\phi_i - \phi_{LOS})]^2 + \sum_{i=1}^{N_h} [y_h(i) - f_h(\phi_i - \phi_{LOS})]^2 \right\}, \quad (13)$$

where $\{y_v(i), \forall i = 1, 2, \ldots, N_v\}$ and $\{y_h(i), \forall i = 1, 2, \ldots, N_h\}$ refer to the empirical data points for the vertically polarised and horizontally polarised receive antennas, respectively. This empirical data set of [18] shows values over only $\phi \in [71^\circ, 92^\circ]$ for the vertically polarised receiving antenna and over only $\phi \in [72^\circ, 91^\circ]$ for the horizontally polarised receiving antenna, instead of over $\forall \phi \in [-180^\circ, 180^\circ]$ as derived in (11) and (12). Hence, these two derived formulas need each be re-normalised to unit area over these more constricted $\phi$-ranges. Likewise, each polarisation’s empirical data set also needs similar amplitude normalisation within these restricted $\phi$-ranges. The parameter $\phi_{LOS}$ is to account for the transmitter–receiver line-of-sight azimuth direction, which is unspecified in [35].

The empirical measurements in [35, Fig. 10], collected in Minami-Senzoku, Otaku, Tokyo, Japan, have this measurement scenario: the transmitter was elevated on the tenth floor of a building surrounded by other buildings about 6–8 m in height. The transmitter was vertically polarised. The receiver was mobile at the street level with vertically polarised and horizontally polarised receive antennas.

Fig. 4 shows that the proposed model provides a good fit to both the vertically polarised receiving antenna’s data (marked
as red circles) and the horizontally polarised receiving antenna’s data (marked as blue asterisks). Here, $a_v = 1$ and $a_h = 0$; and (11) and (12) consequently degenerate to a uni-Gaussian-distributed cluster. These two clusters are mathematically interrelated by the common parameters of $s_{v,v} = N(1)\sqrt{1 - s_{h,v}^2}$ and $\phi_{\text{LOS}}$.

5 Conclusion

This paper has introduced a new tetra-cluster geometric model for the arriving multipaths’ DOA distribution – a model that accounts for the transmit antenna’s and the receiving antenna’s polarisation. This polarisation-sensitive modelling approach is herein demonstrated using Gaussian-distributed clusters for the azimuth-DOA distribution. However, this approach is equally applicable with other cluster distributions, to model the azimuth-elevation DOA distribution or to model the DOA-TOA joint distribution. The proposed model’s efficacy is verified by calibration with limited empirical data.

6 Acknowledgment

This work was supported by the Hong Kong Polytechnic University’s Internal Competitive Research Grant number G.42.R6.YF52.

7 References


Figure 4 Derived azimuth-DOA distributions in (11) and (12), calibrated against empirical data in [18]
Calibrated model parameters are $\Sigma_{v,v}/D = 0.138$, $\Sigma_{v,h}/D = 0.2837$ and $\phi_{\text{LOS}} = 80.788$. 


