Abstract—Aperture extension is achieved in this novel ESPRIT-based two-dimensional angle estimation scheme using a uniform rectangular array of vector hydrophones spaced much farther apart than a half-wavelength. A vector hydrophone comprises two or three spatially co-located, orthogonally oriented identical velocity hydrophones (each of which measures one Cartesian component of the underwater acoustical particle velocity vector-field) plus an optional pressure hydrophone. Each incident source’s directions-of-arrival are determined from the source’s acoustical particle velocity components, which are extracted by decoupling the data covariance matrix’s signal-subspace eigenvectors using the lower dimensional eigenvectors obtainable by ESPRIT. These direction-cosine estimates are unambiguous but have high variance; they are used as coarse references to disambiguate the cyclic phase ambiguities in ESPRIT’s eigenvalues when the intervector-hydrophone spacing exceeds a half-wavelength. In one simulation scenario, the estimation standard deviation decreases with increasing intervector-hydrophone spacing up to 12 wavelengths, effecting a 97% reduction in the estimation standard deviation relative to the half-wavelength case. This proposed scheme and the attendant vector-hydrophone array outperform a uniform half-wavelength spaced pressure-hydrophone array with the same aperture and slightly greater number of component hydrophones by an order of magnitude in estimation standard deviation. Other simulations demonstrate how this proposed method improves underwater acoustic communications link performance. The virtual array interpolation technique would allow this proposed algorithm to be used with irregular array geometries.

Index Terms—Array signal processing, direction of arrival estimation, eigenfunctions/eigenvalues, intelligent sensors, sonar arrays, sonar position measurement, sonar signal processing, underwater acoustics arrays.

I. INTRODUCTION

A. Basic Principles Underlying the New Algorithm

1) The Aperture Extension Problem: An array aperture with enlarged aperture size generally offers enhanced array resolution and direction-finding (DF) precision. Adding more array elements to extend array length would increase hardware costs and would add considerably to the computational load required by the signal processors. Nonuniform intersensor spacing over a larger aperture would generally violate ESPRIT’s (estimation of signal parameters via rotational invariance techniques [8]) prerequisite of two identical but translated subarrays. Extending the uniform intersensor spacing beyond a half-wavelength will lead to a set of cyclically ambiguous direction-cosine estimates [hence, a set of ambiguous direction-of-arrival (DOA) estimates], in accordance with the spatial Nyquist sampling theorem. Without a priori source information, this cyclic ambiguity in the phases of ESPRIT’s eigenvalues cannot be resolved using customary pressure hydrophones. In contrast to the foregoing approaches of aperture extension, this novel algorithm: 1) allows the uniform interelement spacing to exceed the Nyquist half-wavelength upper limit; 2) solves the aforementioned ambiguity problem; and 3) automatically pairs estimates of the direction cosines along the $z$ axis with those along the $y$ axis. This proposed approach achieves all these by extracting DOA information embedded in the vector components of the incident vector wavefield through the use of spatially co-located but orthogonally oriented velocity hydrophones. This vector-component information plus the aforementioned aperture extension bring about increased estimation accuracy and resolution capabilities.

2) Implications for Underwater Acoustic Communications: Spatial diversity improves underwater acoustic communications link performance because of the expectation that if one channel is in a deep fade, the multipaths may be constructive interfering in another channel. Spatial diversity is realized by deploying multiple spatially displaced hydrophones, typically at the base station. In frequency-selective slow-fading environments (slow-fading implies constant fading characteristics over a short observation period), spatial diversity also allows various time-delayed multipaths to be resolved based on their distinct spatial arrival angles. These resolved multipaths may then be constructively summed to maximize the signal-to-noise ratio (SNR). Moreover, estimates of these sources’ up-link arrival angles facilitate down-link beamforming from the base-station to the mobile unit.

3) Vector Hydrophones: This innovative scheme employs a sparse array of identical vector-hydrophones, each of which embodies a pair or a triad of spatially co-located but orthogonally oriented identical velocity hydrophones plus an optional
co-located pressure hydrophone. Each velocity hydrophone measures one Cartesian component of the impinging acoustical particle velocity vector-field. Velocity-hydrophone technology has been available for decades [1] and continues to attract growing attention [28] in the field of underwater acoustics. Many different types of velocity hydrophones are available [2] and have been constructed using a variety of technologies, with designs ranging from mechanically based [4], to optically based [6], to derivative-based [16]. One implementation of the vector hydrophone is the DIFAR sensor [11], [12].

Note that a pair of identical vector hydrophones embody two identical subarrays displaced in space. Thus, as few as only two vector hydrophones would be needed to construct an intersubarray spatial invariance to allow the application of ESPRIT or its total least squares (TLS) variant TLS-ESPRIT [8]. The proposed array geometry (actual or virtual) of this paper places one vector hydrophone on each grid point of an extended but equally spaced grid system. Grid geometries suggested and investigated in this paper include a rectangular array with \( L \times M \) vector hydrophones located at \((l \Delta_x, m \Delta_y)\) for \(l = 1, \ldots, L; m = 1, \ldots, M\) in \((x, y)\) coordinates. These array geometries could be “virtual” arrays interpolated [10], [13] from actual array configurations that differ somewhat from the above specified geometries. The above intervector-hydrophone spacings \(\Delta_x\) and \(\Delta_y\) are set to exceed greatly the Nyquist half-wavelength upper limit to effect a large array aperture, leading to correspondingly more highly accurate source direction estimates. A spatial invariance along the \(x\) axis may be formed by considering the vector hydrophones in the \(x = 1\) to the \(x = L - 1\) rows as one subarray, and those vector hydrophones in the \(x = 2\) to the \(x = L\) rows as another subarray. Application of ESPRIT to the data from these two overlapping subarrays would produce estimates of the direction cosine of each source relative to the \(x\) axis, \(\{\phi_k = \cos \theta_k, k = 1, \ldots, K\}\), where \(\phi_k\) and \(\theta_k\) are the azimuth and elevation arrival angles of the \(k\)th incident signal. Simultaneously, ESPRIT is applied to the data from the two subarrays with \(y\)-axis spatial invariance to produce estimates of the direction cosine of each source relative to the \(y\) axis, \(\{\theta_k = \cos \phi_k \sin \phi_k, k = 1, \ldots, K\}\). This second pair of subarrays may be constructed by taking the first \(M - 1\) and the last \(M - 1\) columns of the \(L \times M\) array of vector hydrophones. In fact, as few as merely three vector hydrophones (in a triangular geometry at \((0, 0), (\Delta_x, 0), (0, \Delta_y)\) in \((x, y)\) Cartesian coordinates) can suffice to implement this novel extended aperture algorithm.

4) Removal of Cyclic Ambiguity: When the intervector-hydrophone spacings, \(\Delta_x\) and \(\Delta_y\), exceed a half-wavelength, there is an ambiguity of some integer multiple of \(2\pi\) in the phase of each eigenvalue generated in the final stage of TLS-ESPRIT. This leads to ambiguous DOA estimates equispaced by \(\lambda/\Delta_x\) in the interval \(-1 \leq \phi_k < 1\) and \(\lambda/\Delta_y\) in the interval \(-1 \leq \theta_k < 1\). There no longer exists any one-to-one mapping between the phase of ESPRIT’s eigenvalue and the corresponding source’s direction cosine. If this ambiguity could be removed, the resulting direction-cosine estimates would be highly accurate due to the sparse array’s large aperture. On the other hand, the signal-subspace eigenvectors of the data covariance matrix do not suffer any similar extended-aperture ambiguity regardless of the intervector-hydrophone spacing. Thus, the key idea of this paper is to use the unambiguous DOA information embedded in the impinging wavefield’s acoustical particle velocity-field components to resolve the cyclic ambiguity in ESPRIT’s eigenvalues’ phases due to extended intervector-hydrophone spacing. The unambiguous DOA information embedded in the impinging wavefield’s vector components may be extracted by using the lower dimensional eigenvectors associated with the coupling matrix (not the signal-subspace eigenvectors nor the noise-subspace eigenvectors of the data correlation matrix) generated in the final stage of TLS-ESPRIT to decouple the element-space signal-subspace eigenvectors to yield the sources’ respective vector-hydrophone array manifolds. These vector-hydrophone array manifold estimates would yield coarse but unambiguous Cartesian direction-cosine estimates because the vector-hydrophone array manifold estimate, when properly normalized, would have as its components the corresponding source’s Cartesian direction cosines. The idea is to use these as “coarse” references to resolve the cyclic ambiguity (equal to some integer multiple of \(\lambda/\Delta\)) in the direction-cosine estimates yielded by ESPRIT’s eigenvalues. The direction-cosine estimates produced by the impinging wavefield’s estimated vector components have been characterized above as unambiguous but high in variance in comparison to the direction-cosine estimates extracted from the TLS-ESPRIT eigenvalues. This may be intuitively explained by the fact that the former are inherently extracted from information based on a single vector hydrophone which has no effective geometric aperture, whereas the latter are extracted from information that encompasses the entire physical aperture. This comparative characterization of the set of coarse reference estimates versus the set of fine but ambiguous estimates is confirmed by simulations to be presented in Section IV.

5) Automatic Pairing of Direction Cosines Along Different Cartesian Coordinates: Arrival angle estimation problems often have two-dimensional (2-D) arrival angles with arbitrary elevations and azimuths. However, there exists no fundamental algebraic theorem or general algorithm for such a 2-D problem. There do exist numerous algorithms that apply only to arrays that separately estimate the two sets of direction cosines and then pair them. For example, using a planar or \(L\)-shaped array, sensors displaced along the \(x\) axis lead to estimates of the direction cosine along the \(x\) axis; and sensors displaced along the \(y\) axis lead to estimates of direction cosines along the \(y\) axis. However, there still remains the nontrivial problem to pair the \(x\)-axis direction-cosine estimates with the \(y\)-axis direction-cosine estimates. In contrast, this pairing problem is automatically solved in the proposed algorithm’s disambiguation step, where the incident wavefield’s estimated vector components serve as coarse references. Because these velocity vector-field estimates have already paired the direction cosines, these estimates can thus, also serve as references to pair the direction-cosine estimates obtained from ESPRIT’s eigenvalues.

Note that the disambiguation and the direction-cosines pairing mentioned above would be impossible with customary
pressure-hydrophone arrays. The signal-subspace eigenvectors of such a pressure-hydrophone array would suffer a phase ambiguity similar to that of its eigenvalues. There would thus be no disambiguation reference available from ESPRIT’s eigenvectors.

B. Summary of Relevant Literature

ESPRIT [8] embodies one of the most popular eigenstructure (subspace) source localization methods developed in recent years. Though suboptimal, eigenstructure methods: 1) require no a priori knowledge of the sources’ joint spectral densities, but only the second-order-statistics of the additive noise; 2) impose lighter computational load; and 3) offer estimation performance comparable to the optimal methods at moderate SNR.

Diverse approaches have been advanced to extend array aperture with a sparse array: 1) Swingler and Walker [7] model impinging signals as ARMA time series, estimate the ARMA coefficients from the data correlation matrix, and then use linear prediction to extrapolate actual data collected at actual array elements for virtual array elements located outside the physical array aperture; 2) Shiue et al. [9], Tufts et al. [18], and Wong and Zoltowski [24], [25] construct arrays with two or more sizes of intersensor spacing between adjacent array elements; and 3) Dogan and Mendel [19] use higher order statistics, which require long observation times and exceedingly intensive computation. The novel algorithm introduced in this paper embodies a fourth approach, recognizing the vector-field nature of the impinging underwater acoustical wavefield and thus exploiting the arrival angle information embedded in individual Cartesian components of the impinging particle velocity vector-field. It would be possible to combine certain conceptual elements in the first three approaches of aperture extension with the present approach for maximum aperture extension.

Co-located but orthogonally oriented velocity hydrophones have been used by D’Spain et al. [12] in linearly constrained minimum-variance (LCMV) beamforming toward predetermined directions. Shchurov et al. [14] had also deployed similar arrays to measure ambient noises but not for source localization. That the normalized array manifolds of a triad of co-located but orthogonally oriented velocity hydrophones contain the sources’ direction cosines as components has been a well-known fact but is reiterated also by Nehorai and Paldi [16], who proposed using such vector hydrophones to estimate the arrival angles. Hawkes and Nehorai [20] adapted the Capon method of spectrum estimation to arrays of velocity hydrophones. Hawkes and Nehorai [29] also investigated the performance of vector hydrophones mounted on rigid-pressure surfaces or pressure-releasing surfaces. Entirely original to this present paper is the adaptation of the normalization estimator to the eigenstructure (subspace) source localization approach and to ESPRIT in particular. The conference version [26] of this paper was the first paper recognizing the possibility of aperture extension in underwater acoustics through such normalization estimators. Wong and Zoltowski [30] also proposed another ESPRIT-based direction-finding algorithm for arbitrarily spaced vector hydrophones at unknown locations. Another direction-finding algorithm with irregularly spaced vector hydrophones (at known locations) that adaptively steers null beams in the underwater acoustic particle velocity vector field and that self-initiates a subsequent iterative search without any a priori source information has been presented by Wong and Zoltowski [31]. Multiple-source direction finding with only a single vector hydrophone (rather than an array of multiple vector hydrophones) is also possible by the method in [27], developed also by Wong and Zoltowski.

This paper is organized as follows. Section II formulates the mathematical data models of the vector hydrophone and a planar array of vector hydrophones. Section III develops this novel vector-hydrophone extended-aperture algorithm in several subsections. Section IV then presents and analyzes simulation results, verifying the efficacy of this novel algorithm and its superior performance over half-wavelength spaced scalar-sensor arrays of comparable array-manifold size and computational load. Section V concludes the entire paper.

II. MATHEMATICAL DATA MODEL

Uncorrelated underwater acoustic plane-waves, having traveled through a homogeneous isotropic medium, impinge upon a planar array of uniformly spaced and identical vector hydrophones. This highly regular geometry may be relaxed using the “virtual array interpolation” technique of [10] and [13]. The kth source is characterized by the $3 \times 1$ velocity-field vector

$$\mathbf{a}_k \equiv \begin{bmatrix} a_{\theta_k} \\ a_{\phi_k} \\ a_{2\theta_k} \end{bmatrix} = \begin{bmatrix} \sin \theta_k \cos \phi_k \\ \sin \theta_k \sin \phi_k \\ \cos \theta_k \end{bmatrix} \quad (1)$$

where $u_k \equiv \sin \theta_k \cos \phi_k$ is the direction cosine along the $x$ axis, $v_k \equiv \sin \theta_k \sin \phi_k$ is the direction cosine along the $y$ axis, $w_k \equiv \cos \theta_k$ is the direction cosine along the $z$ axis, $0 \leq \theta_k < \pi$ denotes the signal’s elevation angle measured from the vertical $z$ axis, and $0 \leq \phi_k < 2\pi$ represents the azimuth angle. Note that the three components of the velocity-hydrophone triad’s array manifold would simply correspond to the three direction cosines of the impinging source. This fact is basic to the efficacy of the present algorithm. Equation (1) represents the vector-hydrophone construction using only three velocity hydrophones.

For the vector-hydrophone construction using three velocity hydrophones plus one pressure hydrophone, the vector-hydrophone array manifold is

$$\mathbf{a}_k \equiv \begin{bmatrix} a_{\theta_k} \\ a_{\phi_k} \\ a_{2\theta_k} \\ 1 \end{bmatrix} = \begin{bmatrix} \sin \theta_k \cos \phi_k \\ \sin \theta_k \sin \phi_k \\ \cos \theta_k \\ 1 \end{bmatrix}. \quad (2)$$

Note that $u_k = \sqrt{1 - v_k^2 - w_k^2}$. For the vector-hydrophone construction using three velocity hydrophones, the vector-hydrophone array manifold is the same as in (1). For the vector-hydrophone construction using two velocity hydrophones plus one pressure hydrophone, the normalized vector-hydrophone array manifold would have two out of the
top three components plus the fourth component in (2). For example, to avoid directly dealing with the vertical component of the underwater acoustical particle motion, the vector hydrophone may comprise only one of the two horizontally oriented velocity hydrophones plus a pressure hydrophone

\[
\mathbf{a}_k \triangleq \begin{bmatrix}
\alpha_{x_k} \\
\alpha_{y_k} \\
\Delta 
\end{bmatrix} = \begin{bmatrix}
\sin \theta_k \cos \phi_k \\
\sin \theta_k \sin \phi_k \\
1 
\end{bmatrix}.
\] (3)

Similarly, for the vector-hydrophone construction using only two velocity hydrophones, the vector-hydrophone array manifold would have two out of the three components in (1). However, only those vector-hydrophone constructions with a pressure hydrophone can resolve sources emitting from \(0 \leq \theta_k < \pi\); the vector-hydrophone construction using only velocity hydrophones but no pressure hydrophone can handle only \(0 \leq \theta_k < \pi/2\) (for reasons to be discussed in Section III-A3.

1) Vector-Hydrophone Array Model: Modeling the underwater acoustic sources as narrow-band in that their bandwidths are each negligibly small compared to the carrier frequency, the spatial phase factor for the \(k\)th incident source to the \((l, m)\)th vector hydrophone located at \((\Delta x, m\Delta y)\) is

\[
q_{l,m}(\theta_k, \phi_k) \triangleq e^{j2\pi\Delta x \sin \theta_k / \lambda} e^{j2\pi m \sin \phi_k / \lambda} = e^{j2\pi (\Delta x \sin \theta_k + m \sin \phi_k) / \lambda}.
\] (4)

The \(k\)th signal impinging upon the \((l, m)\)th vector hydrophone at time \(t\) thus registers the vector measurement

\[
\mathbf{s}(t) \triangleq \begin{bmatrix}
s_1(t) \\
s_2(t) \\
\vdots \\
s_K(t) 
\end{bmatrix} = \mathbf{a}_k \mathbf{w}_k(t) + \mathbf{n}(t),
\] (5)

A. Adapting ESPRIT to a Uniform Rectangular Vector-Hydrophone Array

ESPRIT exploits the translational invariance present in any array composed of two identical sub-arrays displaced by some known displacement \(\Delta\). Two distinct matrix-pencil pairs may

\[
\mathbf{Z} \triangleq \begin{bmatrix}
\mathbf{Z}_{L,1} \\
\mathbf{Z}_{L,2} \\
\vdots \\
\mathbf{Z}_{L,M}
\end{bmatrix} = \begin{bmatrix}
\mathbf{z}_{L,1}(t_1) & \cdots & \mathbf{z}_{L,1}(t_N) \\
\vdots & \ddots & \vdots \\
\mathbf{z}_{L,2}(t_1) & \cdots & \mathbf{z}_{L,2}(t_N) \\
\vdots & \ddots & \vdots \\
\mathbf{z}_{L,M}(t_1) & \cdots & \mathbf{z}_{L,M}(t_N)
\end{bmatrix},
\] (11)

where each of the \(LM\) submatrices \(\mathbf{Z}_{L,m}\) of size \(J \times N\) corresponds to data measured by the \((l, m)\)th vector hydrophone.
be constructed with the uniform $L \times M$ rectangular array configuration. One matrix pencil is formed by considering the $(L - 1) \times M$ vector hydrophones on the first to the $(L - 1)\text{th}$ rows of the rectangular grid of the array layout as one sub-array and the $(L - 1) \times M$ vector hydrophones on the second to the $L\text{th}$ rows of the rectangular grid as the other sub-array. This matrix pencil has a spatial invariance along the $x$ axis and can yield estimates of the direction cosines \{$u_k, k = 1, \ldots, K\}. The second matrix pencil is formed by considering the $L \times (M - 1)$ vector hydrophones on the first to the $(M - 1)\text{th}$ columns in the rectangular grid of the array layout as one sub-array and the $L \times (M - 1)$ vector hydrophones on the second to the $M\text{th}$ rows in the rectangular grid as the other sub-array. This matrix pencil has a spatial invariance along the $y$ axis and can yield estimates of the direction cosines \{$v_i, i = 1, \ldots, K\}.

The first step in ESPRIT is to compute the $K (JLM \times 1)$ signal-subspace eigenvectors by eigen-decomposing the $JLM \times JLM$ data covariance matrix $Z^H$. Although two distinct matrix-pencil pairs have been formed, only one eigen-decomposition of this data covariance matrix is necessary because both matrix pencils are formed from the same $JLM \times N$ data set. Let $E_S$ be the $JLM \times K$ signal subspace matrix composed of the $K$ eigenvectors corresponding to the $K$ largest eigenvalues of the $JLM \times JLM$ sample covariance matrix

$$R_{zz} = Z^H \sum_{n=1}^{N} z(t_n)z^H(t_n) = E_S D_S E_S^H + E_N D_N E_N^H$$

where $E_N$ is the $JLM \times (JLM - K)$ noise subspace matrix composed of the $JLM - K$ eigenvectors corresponding to the $JLM - K$ smallest eigenvalues, $D_S$ is a $K \times K$ diagonal matrix whose diagonal entries are the $K$ largest eigenvalues, and $D_N$ is a $(JLM - K) \times (JLM - K)$ diagonal matrix whose diagonal entries contain the $JLM - K$ smallest eigenvalues. $E_S$ asymptotically approaches the following form:

$$E_S = \mathbf{AT}$$

$$= [q_x(u_1) \otimes q_y(v_1) \otimes a_{11}, \ldots, q_x(u_K) \otimes q_y(v_K) \otimes a_{K}] \mathbf{T}$$

where $\mathbf{T}$ is an unknown $K \times K$ nonsingular matrix to be determined. $\mathbf{T}$ is necessarily nonsingular because both $\mathbf{A}$ and $E_S$ are full-rank matrices.

1) Deriving the Low-Variance But Ambiguous Estimates of $u_k$: For the matrix pencil with spatial invariance along the $x$ axis, define $E_2^H$ as the first $J(L - 1)M$ rows of $E_S$ and $E_2^H$ as the last $J(L - 1)M$ rows of $E_S$

$$E_2^H = \begin{bmatrix} I_{J(L-1)M} \otimes \mathbf{O}_{JM} \end{bmatrix} E_S$$

$$\approx \left[ q_x^{(2)}(u_1) \otimes q_y(v_1) \otimes a_1, \ldots, q_x^{(2)}(u_K) \otimes q_y(v_K) \otimes a_K \right] \mathbf{T}$$

$$\triangleq A_2^2$$

(18)

where

$$q_x^{(2)}(u_k) \triangleq \begin{bmatrix} \exp \left( \frac{j2\pi \Delta_x u_k}{\lambda} \right) \\
\vdots \\
\exp \left( \frac{j2\pi \Delta_x (L - 1) u_k}{\lambda} \right) \end{bmatrix}$$

(19)

and $I_{J(L-1)M}$ is a $J(L-1)M \times J(L-1)M$ identity matrix and $\mathbf{O}_{JM}$ is an $M \times M$ matrix whose elements are all zeroes. Since $q_x^{(2)}(u_k) = e^{j2\pi \Delta_x u_k/\lambda} q_x^{(1)}(u_k)$; thus, $A_2^2 = A_1^2 \Phi^u$, where $\Phi^u$ is a $K \times K$ diagonal matrix with diagonal elements 

$$\{[\Phi^u]_{kk} = e^{j2\pi \Delta_x u_k/\lambda}, k = 1, \ldots, K\}.$$ 

The above approximations are exact identities in the noiseless or asymptotic cases. In noiseless or asymptotic cases, there exists a $K \times K$ nonsingular matrix $\Psi^u$ relating the two $J(L - 1)M \times K$ full-rank matrices $E_1^H$ and $E_2^H$ [4]:

$$E_1^H \Psi^u = E_2^H = A_1^2 \Phi^u \Psi^u = A_1^2 \Psi^u$$

$$\Rightarrow \Psi^u = (E_1^H E_1^H)^{-1} (E_1^H E_1^H)^{-1} = (T^u)^{-1} \Phi^u T^u.$$ (21)

Note that $\{[\Phi^u]_{kk}, k = 1, \ldots, K\}$ constitute the eigenvalues of $\Psi^u$. Because $\Delta_x \gg \lambda/2$ and $-1 \leq u_k \leq 1$, there exists a set of cyclically related candidates for the estimation of $u_k$:

$$n_k(n_u) = \mu_k + n_u \frac{\lambda}{\Delta_x},$$

$$\left| \frac{\Delta_x (1 - \mu_k)}{\lambda} \right| \leq n_u \leq \left| \frac{\Delta_x (1 - \mu_k)}{\lambda} \right|$$

(22)

$$\mu_k = \arg \{[\Phi^u]_{kk}\}, \quad k = 1, \ldots, K$$

(23)

where $\lfloor x \rfloor$ is the smallest integer not less than $x$, $\lceil x \rceil$ is the largest integer not greater than $x$, and $\arg z$ is the principal argument of the complex number $z$ between $-\pi$ and $\pi$. Furthermore, the eigenvector corresponding to the eigenvalue $[\Phi^u]_{kk} = e^{j2\pi \Delta_x u_k}$ is the $k\text{th}$ column of $(T^u)^{-1}$, where

$$T^u = \mathbf{P}^u \mathbf{T}$$

(24)

and $\mathbf{P}^u$ is an unknown permutation matrix. In more realistic cases where noise is present and when only a finite number of snapshots are available, all of the above estimates become only approximate.
2) Deriving the Low-Variance but Ambiguous Estimates of \(v_k\): Similarly, for the matrix pencil with spatial invariance along the \(y\) axis, define \(E_1^y\) and \(E_2^y\)

\[
E_1^y = \begin{bmatrix} I_1 & \cdots & I_1 \\ \vdots & \ddots & \vdots \\ I_1 & \cdots & I_1 \\ \end{bmatrix} E_s
\]  
(25)

where \(A_1^y \overset{\text{def}}{=} \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix} \cdot \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix}^T\)

\[
A_2^y \overset{\text{def}}{=} \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix} \cdot \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix}^T \approx \begin{bmatrix} q_y(u_1) \otimes q_y^{(1)}(u_1) \otimes a_1, \ldots, q_y(u_K) \otimes q_y^{(1)}(u_K) \otimes a_K \end{bmatrix} \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix} \cdot \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix}^T
\]  
(26)

\[
E_2^y = \begin{bmatrix} I_2 & \cdots & I_2 \\ \vdots & \ddots & \vdots \\ I_2 & \cdots & I_2 \\ \end{bmatrix} E_s
\]  
(27)

where \(A_2^y \overset{\text{def}}{=} \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix} \cdot \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix}^T \approx \begin{bmatrix} q_y(u_1) \otimes q_y^{(2)}(u_1) \otimes a_1, \ldots, q_y(u_K) \otimes q_y^{(2)}(u_K) \otimes a_K \end{bmatrix} \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix} \cdot \begin{bmatrix} \lambda \Delta_y^y \end{bmatrix}^T
\]  
(28)

where

\[
q_y^{(1)}(u_k) \overset{\text{def}}{=} \begin{bmatrix} \exp \left( j2\pi \Delta_y^y u_k \right) \\ \vdots \\ \exp \left( j2\pi \Delta_y^y (M-1)u_k \right) \\ \end{bmatrix}
\]

\[
q_y^{(2)}(u_k) \overset{\text{def}}{=} \begin{bmatrix} \exp \left( j2\pi \Delta_y^y 2u_k \right) \\ \vdots \\ \exp \left( j2\pi \Delta_y^y M\Delta_y^y u_k \right) \\ \end{bmatrix}
\]

\[
\hat{I}_1 \overset{\text{def}}{=} \begin{bmatrix} I_{J(M-1)} & \cdots & O_J \\ \vdots & \ddots & \vdots \\ O_J & \cdots & I_{J(M-1)} \\ \end{bmatrix}
\]

\[
\hat{I}_2 \overset{\text{def}}{=} \begin{bmatrix} I_{J(M-1)} & \cdots & O_J \\ \vdots & \ddots & \vdots \\ O_J & \cdots & I_{J(M-1)} \\ \end{bmatrix}
\]

Since \(q_y^{(2)}(u_k) = e^{2\pi \Delta_y^y u_k/\lambda} q_y^{(1)}(u_k)\), \(A_2^y = A_1^y \Phi^y\), where \(\Phi^y\) is a \(K \times K\) diagonal matrix whose diagonal elements \(\{\Phi^y\}_{jj} = e^{(2\pi / \lambda) \Delta_y^y u_j}, j = 1, \ldots, K\). The approximations above are exact identities in the noiseless or asymptotic cases. In noisless or asymptotic cases, there exists a \(K \times K\) nonsingular matrix \(\Phi^y\) relating the two \(J(L-1)M \times K\) full-rank matrices \(E_1^y\) and \(E_2^y\):

\[
E_1^y \Phi^y = E_2^y \Rightarrow A_1^y T^y \Phi^y = A_2^y T^y
\]

\[
= \Phi^y = \{(E_1^y)^H E_1^y\}^{-1} \{E_2^y\} E_2^y
\]

(31)

(32)

The direction cosine of the \(j\)th source relative to the \(y\) axis are

\[
\hat{\nu}_j(n_v) = \nu_j + n_v \frac{\lambda}{\Delta_y^y} \left[ \frac{\Delta_y^y}{\lambda} (1 - \nu_j) \right] \leq n_v \leq \left[ \frac{\Delta_y^y}{\lambda} (1 - \nu_j) \right]
\]

(33)

\[
\nu_j = \arg \left( \frac{\langle \Phi^y \rangle_{jj}}{\lambda} \right), \quad j = 1, \ldots, K.
\]

(34)

Equation (33) represents a set of low-variance but ambiguous estimates of the direction cosine of the \(k\)th source relative to the \(y\) axis. Furthermore, the eigenvector corresponding to the eigenvalue \(\Phi^y\), is the \(k\)th column of \((T^y)^{-1}\), where

\[
T^y = P^y T
\]

(35)

and \(P^y\) is an unknown permutation matrix. In more realistic cases, where noise is present, and when only a finite number of snapshots are available, all of the above estimates become only approximate.

3) Deriving the Unambiguous Coarse Reference Estimates of \(v_k\) and \(v_k\): In order to derive the set of high-variance but unambiguous direction cosines, the vector-hydrophone array manifolds for each of the \(K\) sources needs to be estimated. This may be accomplished by decoupling the data covariance matrix’s signal-subspace eigenvectors as follows. The matrix-pencil pair with \(x\)-axis spatial invariance gives

\[
\hat{A}_k^x = \sum_{l=1}^{L} \sum_{m=1}^{M} \{E_{l,m}^x(T^x)^{-1} + E_{l,m}^x(T^x)^{-1} \Phi^{xk}\}
\]

(36)

where each of the \(LM\) submatrices \(E_{l,m}^x\) of size \(J \times K\) represents that part of \(E^x\) corresponding to the \((l, m)\)th vector hydrophone (and similarly for \(E_{l,m}^y\)), and \(e_k\) denotes a \(K \times 1\) vector with all zeros except a one at the \(k\)th position. The matrix-pencil pair with the \(y\)-axis spatial invariance gives

\[
\hat{A}_k^y = \sum_{l=1}^{L} \sum_{m=1}^{M} \{E_{l,m}^y(T^y)^{-1} + E_{l,m}^y(T^y)^{-1} \Phi^{yk}\}
\]

(37)

where \(E_{l,m}^x\) and \(E_{l,m}^y\) are defined similar to the definitions of \(E_{l,m}^x\) and \(E_{l,m}^y\). The inclusion of \(\Phi^{xk}\) and \(\Phi^{yk}\) in the above expressions ensures coherent summation and is vital to achieving the best possible estimation performance. Note that \(\hat{A}_k^x\) is already paired with \(\mu_k\), and \(\hat{A}_k^y\) is already paired with \(\nu_k\).

However, for any \(T^x\) that satisfies (21) and any diagonal matrix \(T^y\) with unit-magnitude complex scalars on its main diagonal, \(T^x T^y\) would also satisfy (21). The same is true for \(T^y\) and (32). Thus, \(\hat{A}_k^x\) and \(\hat{A}_k^y\) are each an estimate of \(a_k\) within some unimagnitude complex scalar. For the cases where a pressure hydrophone is included in the construction of each vector hydrophone, the \(k\)th element of \(\Phi^{xk}\) must be equal to 1. Hence, the aforementioned unit-magnitude complex scalar ambiguity may be removed as follows:

\[
\tilde{\hat{a}}_k^x \overset{\text{def}}{=} \frac{\hat{a}_k^x}{\left| \hat{a}_k^x \right|}, \quad \tilde{\hat{a}}_k^y \overset{\text{def}}{=} \frac{\hat{a}_k^y}{\left| \hat{a}_k^y \right|}
\]

(38)
However, for the cases where each vector hydrophone is constructed using only velocity hydrophones but no pressure hydrophone, the normalized term \( \hat{A}_V^{i,j} / \hat{A}_V^{j,i} \) would still produce two different possible sets of estimates, because for any \((\theta, \phi)\) that leads to \( \hat{A}_V^{i,j} / \hat{A}_V^{j,i} \) \((\pi - \theta, \pi + \phi)\) would lead to \(-\hat{A}_V^{i,j} / \hat{A}_V^{j,i}\), which also represents a legitimate vector-hydrophone steering vector. Only if either \(\theta\) is restricted to \(0 \leq \theta < \pi/2\) (rather than \(0 \leq \theta < \pi\)) or if \(\phi\) is confined to \(0 \leq \phi < \pi\) (rather than \(0 < \pi < 2\pi\)) would this \(\pm 1\) ambiguity be resolvable. That is, for the three-velocity-hydrophone construction of the vector hydrophone in (1), direction-finding is confined to only one hemisphere of the three-dimensional space surrounding the array.

Thus far, unambiguous but high-variance estimates \(\{\hat{a}_1^{i,j}, \hat{a}_2^{i,j}, \hat{a}_3^{i,j}\} \) for \(\{u_h, v_h, w_h\}\) have been found from \(\hat{A}_V^{i,j}\); and likewise from \(\hat{A}_V^{j,i}\), another set of unambiguous but high-variance estimates \(\{\hat{a}_1^{i,j}, \hat{a}_2^{i,j}, \hat{a}_3^{i,j}\} \) for \(\{u_j, v_j, w_j\}\) have also been determined. Note that \(\{\hat{a}_1^{i,j}, \hat{a}_2^{i,j}, \hat{a}_3^{i,j}\} \) are already paired with \(\mu_k\), and \(\{\hat{a}_1^{j,i}, \hat{a}_2^{j,i}, \hat{a}_3^{j,i}\} \) are likewise already paired with \(\mu_j\).

Note also that different indices are used to enumerate the coarse reference estimates \(\{\hat{a}_1^{i,j}, \hat{a}_2^{i,j}, \hat{a}_3^{i,j}\} \) and \(\{\hat{a}_1^{j,i}, \hat{a}_2^{j,i}, \hat{a}_3^{j,i}\} \) as well as the eigenvalues of \(\hat{\Psi}_V\) and \(\hat{\Psi}_V\) (i.e., the low-variance but cyclically ambiguous estimates of the direction cosines \(\{\hat{v}_k, k = 1, \ldots, K\}\) and \(\{\hat{v}_j, j = 1, \ldots, K\}\). This is because even though \(\hat{\Psi}_V\) and \(\hat{\Psi}_V\) have common eigenvectors (in the noiseless or asymptotic cases), the ordering of the eigenvectors of \(\hat{\Psi}_V\) will generally be permuted relative to the ordering of the eigenvectors of \(\hat{\Psi}_V\) due to the presence of the unknown permutation matrices \(\hat{P}_V\) and \(\hat{P}_V\) in (24) and (35). However, \(\{\hat{a}_1^{i,j}, \hat{a}_2^{i,j}, \hat{a}_3^{i,j}\}\) and \(\{\hat{a}_1^{j,i}, \hat{a}_2^{j,i}, \hat{a}_3^{j,i}\}\) can be easily paired as follows:

\[
\hat{a}_k^{i,j} = \arg \min \|[\hat{a}_1^{i,j}, \ldots, \hat{a}_K^{i,j}] - [\hat{a}_1^{j,i}, \ldots, \hat{a}_K^{j,i}]\|.
\]

(39)

The above minimization is with respect to all possible permutations of \(\{k_1, \ldots, K\}\). From \(\hat{a}_k^{i,j}\) and \(\hat{a}_k^{j,i}\), an overall \(\hat{a}_k\) may thus be formed:

\[
\hat{a}_k = \left[\begin{array}{c}
\hat{a}_k^{i,j} \\
\hat{a}_k^{j,i}
\end{array}\right] = \frac{\hat{a}_V^{i,j} + \hat{a}_V^{j,i}}{\|\hat{A}_V^{i,j} + \hat{A}_V^{j,i}\|}.
\]

(40)

\(\{\hat{a}_V^{i,j}, \hat{a}_V^{j,i}, \hat{a}_V^{k,j}\}\) are already paired with \(\mu_k\) and \(\{\hat{a}_V^{j,i}, \hat{a}_V^{k,j}, \hat{a}_V^{i,j}\}\) are likewise already paired with \(\mu_j\). Recall that \(\mu_k\) is the principal argument of the \(k\)th eigenvalue of \(\hat{\Phi}_V\), it follows that \(\{\mu_1, \ldots, \mu_K\}\) is to be paired with \(\{\nu_1, \ldots, \nu_K\}\), in that order.

For the case where each vector hydrophone in the array is constructed with three velocity hydrophones plus a pressure hydrophone, additional noise cancellation is possible using the Lagrange multiplier method to harmonize the two separate measurements, \(\sqrt{2} |\hat{a}_V^{i,j}|^2\) and \(\sqrt{2} (|\hat{a}_V^{i,j}|^2 + |\hat{a}_V^{j,i}|^2 + |\hat{a}_V^{k,j}|^2)^2\), of the normalized pressure intensity. An optimal set of perturbation weights \(w_1, w_2, w_3, w_4\) relating the elements of \(\hat{a}_k\):

\[
(1 + w_h) |\hat{A}_V^{i,j}|^2 = \frac{\sum_{i=1}^{4} (1 + w_h) |\hat{A}_V^{i,j}|^2}{|\hat{A}_V^{i,j}|^2}.
\]

(41)

may be derived using the Lagrange multiplier method. Toward this end, recast the problem to the conceptual framework of optimization: \(w_1^2 + w_2^2 + w_3^2 + w_4^2\) is to be minimized given the constraint in (41). The optimal perturbation weights are then

\[
w_1 = -\lambda |\hat{A}_V^{i,j}|_j \\
w_2 = -\lambda |\hat{A}_V^{j,i}|_j \\
w_3 = -\lambda |\hat{A}_V^{k,j}|_j \\
\lambda = \frac{\sum_{i=1}^{4} (|\hat{A}_V^{i,j}| - |\hat{A}_V^{i,j}|_j)^2}{\sum_{i=1}^{4} (|\hat{A}_V^{i,j}|)^2 - (|\hat{A}_V^{i,j}|_j)^2}.
\]

(42)

(43)

(44)

Thus, the improved coarse reference direction-cosine estimates are

\[
\hat{a}_k^{i,j} = \sqrt{2} (1 + w_h) |\hat{A}_V^{i,j}|_j \\
\hat{a}_k^{j,i} = \sqrt{2} (1 + w_h) |\hat{A}_V^{j,i}|_j \\
\hat{a}_k^{k,j} = \sqrt{2} (1 + w_h) |\hat{A}_V^{k,j}|_j.
\]

(46)

(47)

(48)

Because it would always be necessary to have at least three co-located hydrophones at each grid point so as to normalize the direction-cosine estimates properly, the MUSIC-based disambiguation step would be necessary for the case where only each vector hydrophone in the array comprises only two velocity hydrophones.

4) Disambiguation of ESPRIT Eigenvalues: The disambiguated estimates are

\[
\hat{v}_k = \mu_k + n_{i_k} \frac{\lambda}{\Delta x} \\
\hat{v}_k = \mu_k + n_{i_k} \frac{\lambda}{\Delta y}
\]

(49)

where \(n_{i_k}\) and \(n_{i_k}\) may be separately estimated as

\[
n_{i_k} = \arg \min \left| \hat{a}_k^{i,j} - \mu_k - n_{i_k} \frac{\lambda}{\Delta x} \right| \\
n_{i_k} = \arg \min \left| \hat{a}_k^{j,i} - \mu_k - n_{i_k} \frac{\lambda}{\Delta y} \right|
\]

(50)

Note that \(n_{i_k}\) and \(n_{i_k}\) are determined separately. The search range for \(n_{i_k}\) and \(n_{i_k}\) in both cases is given by the far right-hand side of (22) and (23), respectively; up to a maximum of \(2(\Delta x/\lambda) + 1\) and \(2(\Delta y/\lambda) + 1\) candidates are tested.

5) Expressions for the Azimuth and Elevation Angle Estimates: From the direction-cosine estimates derived above, the \(k\)th signal’s azimuth and elevation arrival angles may be estimated as \(\hat{\theta}_k = \sin^{-1} \sqrt{\hat{a}_k^{i,j} + \hat{a}_k^{j,i}} \cos^{-1} \hat{a}_k^{i,j} = \tan^{-1} (\hat{a}_k^{i,j}/\hat{a}_k^{j,i})\).

B. Example: Right Triangular Array with Three Vector Hydrophones

In two-angular-dimensional direction finding, it is necessary to have two distinct spatial invariances, one along each dimension. Because any two vector hydrophones represents a pair of identical \(J\)-element subarrays, the smallest vector-hydrophone array to which 2-D ESPRIT is applicable would be
a triangular array of three hydrophones. Such a vector-hydrophone triad can resolve up to $J-1$ uncorrelated sources.

This present subsection will derive the equations for such a right-triangular array of three vector hydrophones by simplifying the mathematics developed in the preceding subsection for the $L \times M$ rectangular array of vector hydrophones. The results in this subsection may easily be extended for cross-shaped or L-shaped arrays with multiple vector hydrophones on each leg.

Given a right-triangular array of three vector hydrophones at $(0,0)$, $(\Delta x, 0)$, and $(0, \Delta y)$ the $(x, y)$-coordinates, the exposition given in the previous section still holds, if modified as follows. The $3J \times 1$ snapshot vector at time $t$ is now expressed as

$$
\mathbf{z}(t) = \begin{bmatrix} 
\mathbf{z}_1(t) \\
\mathbf{z}_2(t) \\
\mathbf{z}_3(t)
\end{bmatrix} = \sum_{k=1}^{K} s_k(t) \mathbf{q}(u_k, v_k) \otimes \mathbf{a}_k + \mathbf{n}(t),
$$

The first, middle, and last $J \times 1$ subvectors of $\mathbf{z}(t)$ are the outputs of the $J$-component hydrophones comprising the vector hydrophone located at the Cartesian coordinates $(0,0)$, $(\Delta x, 0)$, and $(0, \Delta y)$, respectively. $\mathbf{q}(u_k, v_k)$ is the $J \times 1$ propagation phase-offset vector

$$
\mathbf{q}(u_k, v_k) = \begin{bmatrix} 
\exp \left( j \frac{2 \pi}{\lambda} \Delta x u_k \right) \\
\exp \left( j \frac{2 \pi}{\lambda} \Delta y v_k \right)
\end{bmatrix}.
$$

The signal-subspace eigenvector matrix $\mathbf{E}_S$ now becomes

$$
\mathbf{E}_S = \begin{bmatrix} 
\mathbf{E}_1 \\
\mathbf{E}_2 \\
\mathbf{E}_3
\end{bmatrix} = \mathbf{A} \mathbf{T} = \begin{bmatrix} 
\mathbf{A}_1 \\
\mathbf{A}_1 \Phi^u \\
\mathbf{A}_1 \Phi^v
\end{bmatrix} \mathbf{T} \tag{53}
$$

where $\mathbf{A}_1$ is defined as the $J \times K$ matrix

$$
\mathbf{A}_1 = \begin{bmatrix} 
\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_K
\end{bmatrix}.
$$

Next, follow the procedure already explained in the previous section with $\mathbf{E}_1^u \equiv \mathbf{E}_1$, $\mathbf{E}_2^u \equiv \mathbf{E}_2$, $\mathbf{E}_1^v \equiv \mathbf{E}_1$, $\mathbf{E}_2^v \equiv \mathbf{E}_3$, $L_1 \equiv 2$, $M_1 \equiv 2$.

C. Problem of Near-Identical Eigenvalues

When a matrix pencil has equal or near-identical ESPRIT eigenvalues, no unique coupling matrix $\mathbf{T}$ exists. This means that the matrix pencil’s eigenvectors cannot be decoupled to estimate the underwater acoustic velocity-field vectors for the integer search disambiguation procedure. A matrix pencil can have near-identical ESPRIT eigenvalues even if all sources have distinct and well-separated direction cosines. This is because the matrix pencil’s generalized eigenvalues are phase-wrapped when the intervector-hydrophone spacing exceeds a half-wavelength. Thus, even if $u_1 \neq u_2$, but if $u_i = u_2 \pm (n_i \lambda / \Delta x)$ (where $n_i$ is any integer), then $\mathbf{Q}^{2\pi n_i \lambda / \Delta x} \equiv \mathbf{Q}_{\Delta x / \lambda}$. When only one matrix pencil has near-identical generalized eigenvalues, there exists a very simple remedy. Suppose this problem occurs at the matrix pencil along the $x$ axis affecting $\{\mathbf{E}_1^u, \mathbf{E}_2^u\}$. This means there exists no unique $\mathbf{T}^u$ and the array manifolds cannot be recovered from the signal-subspace eigenvectors of the matrix pencil $\{\mathbf{E}_1^u, \mathbf{E}_2^u\}$. However, a unique $\mathbf{T}^v$ still exists for the second matrix pencil $\{\mathbf{E}_1^v, \mathbf{E}_2^v\}$. Thus, $\mathbf{T}^v$ can simply substitute in place of $\mathbf{T}^u$ in (36) to process the first matrix-pencil’s eigenvectors. This works because under noiseless conditions $\mathbf{T}^u \approx \mathbf{T}^v$.

D. MUSIC-Based Disambiguation

MUSIC (MUltiple SIgnal Classification) [3] embodies an alternate eigenstructure method that offers better estimation accuracy in comparison to ESPRIT. After the overall element space is decomposed into a $K$-dimensional signal subspace and a $(LM - K)$-dimensional noise subspace, MUSIC constructs a null spectrum parameterized by the signal parameters. Iterative search over this null spectrum for its $K$ deepest nulls would produce estimates of the $K$ sources’ parameters. This section, however, will present a closed-form adoption of MUSIC’s null spectrum to improve the ESPRIT-based vector-hydrophone extended-aperture direction-finding method developed thus far. This procedure is applicable to any of the four vector-hydrophone constructions presented earlier.

MUSIC’s null spectrum is constructed based on the orthogonality between the signal subspace and the noise subspace of the sample correlation matrix $\mathbf{R}_{ZZ}$. That is, $\mathbf{E}_S \perp \mathbf{E}_N$. Because $\mathbf{E}_S$ and $\mathbf{A}$ span the same column space, $\mathbf{A} \perp \mathbf{E}_N$:

$$
\{ \hat{u}_k, \hat{v}_k \} = \arg \min_{\{u, v\}} \left| \mathbf{E}_N^H [\mathbf{q}_x(u) \otimes \mathbf{q}_y(v) \otimes \mathbf{a}(u, v)] \right| \tag{55}
$$

While the MUSIC algorithm may be applied to arrays of any geometry, the uniformly spaced configuration used in the present algorithm reduces the support region of MUSIC’s iterative search from a continuous region $\{-1 \leq u, v \leq 1\}$ to a small finite set of discrete points

$$
\left\{ \hat{u}_k(n_u), \hat{v}_k(n_v) \right\} \text{ for } \left[ \frac{\Delta x}{\lambda}(1 - \mu_k) \right] \leq n_u \leq \left[ \frac{\Delta x}{\lambda}(1 - \mu_k) \right], \left[ \frac{\Delta y}{\lambda}(1 - \mu_k) \right] \leq n_v \leq \left[ \frac{\Delta y}{\lambda}(1 - \mu_k) \right]. \tag{56}
$$

Using the coarse estimates $\{\hat{u}_{k_1}, \hat{v}_{k_2}\}$, the above range may be further reduced by constricting $n_u$ and $n_v$ such that

$$
|\hat{u}_k(n_u) - \hat{u}_{k_1}| < \frac{\lambda}{\Delta x} N_u \tag{57}
$$

where $N_u$ and $N_v$ are some preset numbers such as 3 or 4.

Note that this MUSIC null spectrum step does not improve the accuracy of either the coarse estimates or the fine estimates. Rather, this MUSIC step realizes more accurate disambiguation.
Fig. 1. RMS standard deviation of $\hat{\theta}_1$ and $\hat{\phi}_1$ versus intervector sensor spacing.

Note that this MUSIC-based disambiguation step renders it unnecessary to use the normalization estimator to derive any unambiguous coarse reference estimates. Thus, it becomes no longer necessary to have available all three velocity-field components of the impinging wavefront. It would be sufficient to have available as few as only any two of the three velocity-field components, because the resulting array manifold would still retain its one-to-one relation with the impinging source’s direction cosine. The cyclically ambiguous set of direction cosines along the $x$ axis may be aligned with the set along the $y$ axis by methods such as that presented in [25]. However, the preceding ESPRIT-based step is still useful in constraining the estimation candidate set from the infinitely many possible values $\{-1 \leq u, v < 1\}$ to the few candidates in (57). Without this narrowing of the candidate field by the preceding ESPRIT-based step, the present MUSIC-based step would have to perform an open-ended iterative search over the highly nonlinear scalar function (55) instead of the much simpler closed-end assessment of the few candidates in (57). Moreover, whether MUSIC converges to the global optimum or mistakenly to a local minimum and the rate that MUSIC converges both depend on the availability of good initial estimates to start off MUSIC’s search. Without the preceding ESPRIT-based step, no such coarse estimates of the direction cosines would be available to MUSIC to start off its iterative search.

IV. SIMULATIONS

Simulations were conducted to verify the efficacy of the extended aperture vector-hydrophone array concept and attendant ESPRIT-based 2-D angle estimation algorithm. The vector hydrophones used in all simulations each comprises three co-located but orthogonally oriented velocity hydrophones. The parameters for the simulation results presented in Figs. 1 and 2 are as follows. An $8 \times 8$ rectangular array of uniformly spaced identical vector hydrophones was used. The signal scenario involved two closely spaced equipowered uncorrelated narrow-band sources from the far field with the following location parameters: $\theta_1 = 80^\circ, \phi_2 = 30^\circ, \phi_2 = 75^\circ, \phi_2 = 35^\circ$. This means that $u_1 = 0.8529, v_1 = 0.4924, u_2 = 0.7912, v_2 = 0.5540$. The SNR per velocity hydrophone is 0 dB for either source. There are 100 snapshots in each independent Monte Carlo simulation experiment and 300 independent experiments per data point. The simulations here implemented the total least squares formulation [7] of ESPRIT for maximum noise cancellation.

For each different value of the intervector-hydrophone spacing $\Delta_x = \Delta_y = \Delta$, sample biases and sample variances for both $\hat{u}_k$ and $\hat{\theta}_k, k = 1, 2$, were computed for each of the two sources from 300 independent trial runs. The rms standard deviation was computed by taking the square root of the sum of the respective samples variances for $\{\hat{u}_1, \hat{\theta}_1, \hat{u}_2, \hat{\theta}_2\}$. An rms bias was also computed by taking the square root of the sum of the respective sample biases $\{\hat{u}_1, \hat{\theta}_1, \hat{u}_2, \hat{\theta}_2\}$. Fig. 1 shows the rms standard deviation as $\Delta$ is increased from a half-wavelength to 64 half-wavelengths. On a log–log scale it is observed that the rms standard deviation decreases linearly as the inter-vector-hydrophone spacing is increased from a half-wavelength up to 12 half-wavelengths, at which point there is a factor of 31 reduction relative to the rms standard deviation obtained with half-wavelength intervector-hydrophone spacing. The rms bias plotted in Fig. 2 exhibits a similar behavior and is more than an order of magnitude lower than the rms standard deviation.

Figs. 1 and 2 also show the rms standard deviation and rms bias of the direction-cosine estimates obtained by the velocity-field vector normalization estimator. Both curves are relatively constant as the intervector-hydrophone spacing is increased from $\Delta = 0.5\lambda$ to $\Delta = 6\lambda$. Both curves are also well above the corresponding curves associated with the disambiguated direction-cosine estimates, with the separation between the two widening as the intervector-hydrophone spacing is increased from $\Delta = 0.5\lambda$ to $\Delta = 6\lambda$. This substantiates the claim that the velocity-field estimation procedure yields unambiguous but high-variance direction-cosine estimates in comparison to the direction-cosine estimates extracted from TLS-ESPRIT’s eigenvalues by (22) and (33). Intuitively, the lower variance
of the latter relative to the former is due to the fact that the
direction cosine estimates extracted from the TLS-ESPRIT’s
eigenvalues depend on the size of the array aperture as
measured by $\Delta$: the larger $\Delta$, the smaller the variance of
$L\hat{\mu}_k/2\pi\Delta$. The relative high variance of the direction cosine
estimates extracted from the velocity-field vector estimates is
intuitively due to the fact that they are inherently extracted
from information provided by a single vector hydrophone,
which has no effective geometric aperture.

In Figs. 1 and 2, a breakdown phenomenon commences
at intervector-hydrophone spacing of $\Delta = 7\lambda$ (14 half-
wavelengths). This breakdown phenomenon may be intuitively
explained by referring to (49)—$\hat{\mu}_k$ and $\hat{\mu}_k$ suffer ambiguities
equal to some unknown integer multiple ($n^\alpha_\mu$ and $n^\beta_\mu$) of
$\lambda/\Delta_x$ and $\lambda/\Delta_y$, respectively. As $n_x$ and $n_y$ vary, new
candidates $\hat{\mu}_k(n_x)$ and $\hat{\mu}_k(n_y)$ emerge and range over the
real number line from $-1$ to 1. In essence, then, the range
of $\hat{\mu}_k(n_x)$ and $\hat{\mu}_k(n_y)$ (i.e., the 2-D rectangle from $-1$ to 1 on
each dimension) becomes segmented by a uniform rectangular
grid system with grid size equal to $2\lambda/\Delta_x \times 2\lambda/\Delta_y$. Each grid point represents one value of $\hat{\mu}_k(n_x)$ and $\hat{\mu}_k(n_y)$.
The disambiguation procedure thus attempts to identify the
exact grid point where the candidates $\hat{\mu}_k(n_x)$ and $\hat{\mu}_k(n_y)$ are
nearest to the to the coarse references $\hat{\mu}_k$ and $\hat{\mu}_k$ obtained
from ESPRIT’s eigenvectors. However, as the inter-vector-
hydrophone spacings $\Delta_x$ and $\Delta_y$ increase, the grid sizes
$2\lambda/\Delta_x$ and $2\lambda/\Delta_y$ shrink relative to the variance of $\hat{\mu}_k$
and $\hat{\mu}_k$. Hence, it becomes increasingly probable that the
coarse references would identify the wrong grid point. As
the inter-vector-hydrophone spacings continue to increase, grid
misidentification will become the dominant error, and both sets
of direction-cosine estimates would have essentially the same
error statistics.

Defining the breakdown inter-vector-hydrophone spacing
to be where the standard deviation of the estimates from
ESPRIT’s eigenvalues become equal to the standard deviation
of the estimates from ESPRIT’s eigenvectors. In Fig. 1, the
breakdown spacing occurs at 16 half-wavelengths, correspond-
ting to a half-grid size of 0.06. (Note that a deviation of plus
or minus a half-grid size from the true value constitutes the
correct grid segment.) This is about five times the coarse
reference estimates’ standard deviation. This supports the
forgoing intuitive analysis of the breakdown phenomenon.
Modeling the coarse reference as a Gaussian random variable,
when the coarse reference’s standard deviation reaches about
1/5 of the half-grid size at the breakdown point, a very few of
the 300 Monte Carlo trials would end in grid misidentification.
As the sample variances and biases are highly sensitive to even
a few outliers, the error statistics are thus skewed upwards.
At the full breakdown spacing, when the coarse reference’s
standard deviation reaches about 2/3 of the half-grid size,
13% of all Monte Carlo trials result in grid misidentification,
and the two sets of direction-cosine estimates (from ESPRIT’s
eigenvectors and from ESPRIT’s eigenvalues) merge in their
error statistics.

Figs. 1 and 2 also demonstrate the superior performance of
this novel extended aperture vector-hydrophone approach over
customary pressure-hydrophone methods. Simulation results
using a 14 x 14 half-wavelength regularly spaced square
array of pressure hydrophones for comparison. This results
in an array aperture of roughly 6.5 wavelengths along either
axis and a 196 x 1 array manifold (comparable to the
192 x 1 array manifold of the 8 x 8 extended-aperture vector-
hydrophone array). Employing the same signal parameters
previously presented, the pressure-hydrophone array yielded
an RMS standard deviation of $1.4 \times 10^{-3}$. In contrast, the
extended-aperture vector-hydrophone array and the attendant
algorithm achieved this same rms standard deviation at $\Delta \approx
0.45\lambda$ (i.e., an aperture of 3.2 wavelengths), i.e., about half
the aperture needed by the half-wavelength-spaced pressure-
hydrophone array! This performance gain is due to the
directionality inherent in vector hydrophones. The performance
gain offered by this innovative algorithm becomes even more
dramatic as $\Delta$ is increased further. Spaced at six wavelengths,
the extended-aperture vector-hydrophone array reduces the
pressure-hydrophone array’s estimation standard deviation by
one whole order of magnitude. This demonstrates the very
impressive performance gain offered by aperture extension. It
would be infeasible to increase the interelement spacing of the pressure-hydrophone array because the ambiguities thus induced are irresolvable.

Simulation results using the additional MUSIC step have also been plotted in Figs. 1 and 2. This additional MUSIC step allows the present algorithm to be valid up to an intervector-sensor spacing of 30 half-wavelengths, instead of 12 half-wavelengths without the MUSIC step.

Figs. 3–6 demonstrate the superior performance of the proposed algorithm in the context of fading-channel wireless mobile communications. One BPSK source, modulated by the raised-cosine waveform with $\beta = 0.35$, impinges upon an $8 \times 8$ vector-hydrophone array with intervector-hydrophone spacing equal to eight half-wavelengths. The channel is slow-fading and frequency-selective, producing two time-delayed multipaths. These two correlated multipaths impinge from the direction cosines $\{u_1 = 0.61, v_1 = 0.79\}$ and $\{u_2 = 0.69, v_2 = 0.71\}$. The second multipath is 3 dB smaller in signal power than the first multipath. Spatial smoothing [5] is applied—four identical but spatially displaced $7 \times 7$ vector-sensor subarrays are formed from the overall $8 \times 8$ vector-sensor array. For comparison, the same signal and channel scenario is simulated for the $14 \times 14$ half-wavelength-spaced pressure-hydrophone array; and spatial smoothing is performed by forming four identical but displaced $13 \times 13$ subarrays. In each case, the observation period is 50 symbols and the sampling frequency is 10 samples per symbol period. One hundred independent Monte Carlo trials are performed for each data point.

In Figs. 3 and 4, the two time-delayed multipaths have relative timing offset equal to 0.3 of a symbol period. The proposed vector-hydrophone extended-aperture algorithm clearly offers significantly lower estimation standard deviations and biases than the half-wavelength-spaced pressure-hydrophone array.
array at all SNR values. Because the two multipaths’ direction cosines are separated by 0.08, the two multipaths may be considered as resolved when their estimation standard deviations fall below 0.02. Using this criteria, the proposed scheme has a resolution threshold at $-32$ dB compared to $-21$ dB for the half-wavelength-spaced pressure-hydrophone array. In the SNR $>-10$ dB range where the standard deviation and bias of both arrays fall off rather linearly with increasing SNR, the proposed method offers an impressive 85% reduction in standard deviation and bias.

In Figs. 5 and 6, the SNR is set at 0 dB and the two time-delayed multipaths’ relative timing offset is varied. Again, the proposed vector-hydrophone extended-aperture algorithm clearly offers significantly lower estimation standard deviations and biases than the half-wavelength-spaced pressure-hydrophone array at all timing offsets. The proposed scheme’s performance remains excellent as the multipath delay decreases, but the half-wavelength-spaced pressure-hydrophone array’s performance denigrates very significantly.

The proposed scheme outperforms the half-wavelength-spaced pressure-hydrophone array, offering a 73%–97% reduction in standard deviation and a 72%–96% reduction in bias.

V. CONCLUSION

The novel algorithm introduced above enlarges the array aperture, but needs no additional sensors, requires no nonuniform interelement spacing, and altogether avoids the direction-cosine ambiguity that commonly arises when interelement spacing exceeds the Nyquist half-wavelength upper limit. Aperture extension is achieved by spacing the vector hydrophones much greater than half-wavelength. Disambiguation of the resulting cyclic ambiguity in the intervector-hydrophone phase factors are accomplished by the use of arrival angle information embedded in each vector-hydrophone’s individual manifold. The success of this innovative method hinges on the fresh insight that the arrival angles may be
estimated from the arrival angle information embedded in each vector-hydrophone’s individual manifold as well as from the intervector-hydrophone phase factors. This innovative approach may be implemented using a variety of array geometries; the specific cases of the rectangular array configuration and the right-triangular configuration have been discussed in detail. The well-known (but much overlooked) vector normalization estimator becomes adopted to ESPRIT source localization in this pioneering paper. The electromagnetic counterpart of this extended-aperture sonar source localization method has also been proposed by the authors [22], [23]. The superior estimation accuracy and SNR resolution threshold of this ground-breaking method over the customary half-wavelength-spaced pressure-hydrophone array approach are credited to the exploitation of the intrinsic directionality of the vector hydrophone in addition to the realization of an extended aperture. Although the proposed algorithm is herein presented in the batch processing mode, real-time adaptive implementations of this present algorithm may be readily realized for nonstationary environments using the fast recursive eigen-decomposition updating methods such as those in [15] and [17]. Furthermore, this algorithm may also be applied to nonuniformly spaced nonrectangular array geometries using the “virtual array interpolation” technique of [10] and [13].

REFERENCES


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