Spatial-Polarizational Correlation-Coefficient Function Between Receiving-Antennas in Radiowave Communications — Geometrically Modeled, Analytically Derived, Simple, Closed-Form, Explicit Formulas

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Abstract—This paper analytically derives closed-form expressions of the uplink received-signal’s polarization-parameterized spatial-correlation-coefficient functions across the base-station antenna-array’s spatial aperture, based on a geometrical model of idealized spatial relationships among the transmitter, the scatterers, and the receiving antennas. The derived formulas fit well with some empirical data.

Index Terms—Communication channels, dispersive channels, fading channels, geometric modeling, multipath channels, scatter channels, spatial correlation.

I. INTRODUCTION

"GEOMETRIC modeling" inter-relates the fading-channel’s various measurable metrics (e.g., the spatial correlation) by abstracting the propagation channel’s spatial geometric relationships among the transmitter, the scatterers, and the receiver – such that this geometric idealization’s very few independent parameters will inter-connectedly affect the various measurable fading metrics to conceptually reveal the channel’s macroscopic statistical structure.

The “spatial correlation” here is across the base-station antenna-array’s physical aperture, as opposed to a correlation over the space traversed by a mobile transmitter, over real time, over the propagation delay, or across frequency.

The system development of “space diversity” or “direction-of-arrival (DOA) diversity” smart-antennas schemes or multiple-input multiple-output (MIMO) schemes can benefit from simple rules-of-thumb that estimate this spatial-correlation-coefficient function.

II. THE PROPOSED STATISTICAL MODEL

A mobile, located at \( z_{MS} = [z_{MS,x}, z_{MS,y}] \) on a two-dimensional Cartesian plane \( \mathbb{R}^2 \), emits a fully polarized signal. This emitted signal bounces off each scatterer in parallel, before reaching the base-station at \( z_{BS} = (0, 0) \). Each scatterer here acts as a re-transmitter, producing one multipath towards each receiving-antenna at the base-station. That is, if \( S \) number of scatterers exist in \( \mathbb{R}^2 \), then \( S \) multipaths will travel from the transmitter to each receiving antenna, with each such multipath representing one “bounce” off a different scatterer out of the \( S \) scatterers. This reflected ray’s polarization and power depend on the scatterer’s intrinsic properties (to be defined momentarily) and the polarization of the incident ray. The free-space path-loss is to be overlooked.

A. The Scatterer as a Polarization-Sensitive Re-Transmitter

From basic electromagnetics, any fully polarized electromagnetic wave may be decomposed as a sum of a vertically polarized component and a horizontally polarized component. When this fully polarized electromagnetic ray reflects (scatterers) off a surface (a scatterer), the reflected (scattered) ray will take on a polarization state that depends on the incident wave’s polarization state, frequency and incident angle, as well as depends on the reflector’s (scatterer’s) electromagnetic properties and surface roughness. More mathematically, let a unit-power incoming ray have vertically polarized power \( a^2_v \) and horizontally polarized power \( a^2_h \). Let the reflected ray have vertically polarized power \( b^2_v \) and horizontally polarized power \( b^2_h \). The relationship among these is governed by the \( 2 \times 2 \) scatterer’s scattering matrix \( S \):

\[
\begin{bmatrix}
    b_v \\
    b_h
\end{bmatrix}
= S
\begin{bmatrix}
    a_v \\
    a_h
\end{bmatrix}
\]

By definition, \( s^2_{v,v} + s^2_{h,v} = 1, s^2_{v,h} + s^2_{h,h} = 1 \), and all elements in \( S \) are non-negative real numbers not larger than 1.

B. The Scatterers’ Spatial Distribution

The scatterers’ spatial density and locations are modeled as a two-dimensional heterogeneous Poisson spatial point process \( \Pi(B) \) indexed on subsets \(^1\) of \( \mathbb{R}^2 \). For any \( B \), the random number \( \Pi(B) \) of points in \( B \) is distributed according to a Poisson law with parameter \( \Lambda(B) = |\Pi(B)| \), the expected number of scatterers in the set \( B \). The present model specifies only the statistical expectation of the scatterer-field’s spatial density, not the spatial density itself. Such a Poisson model can adapt to the field scatterers’ irregular and random spatial variability from one field location to another.

The present work models \( \Pi(B) \) as a sum of two components, \( \Pi_v(B) \) and \( \Pi_h(B) \). The first component \( \Pi_v(B) \) models a cluster of scatterers, each of which would respond to an incoming ray only if it is vertically polarized. If the incident ray is horizontally polarized, any scatterer in \( \Pi_h(B) \) would produce no reflected ray. If an incoming ray consists of a vertically polarized component \( a_v \) and a horizontally polarized component \( a_h \), the scatterer in \( \Pi_v(B) \) will re-transmit only the vertically polarized component, \( a_v \). This re-transmitted ray,

\(^1\) \( B \) belongs to a Borel algebra of sets. That is, \( B \) results from a countable number of set-theory operations with rectangle in the plane \( \mathbb{R}^2 \).
however, will have a vertical component $b_{v,v}$ and a horizontal component $b_{h,v}$. In matrix notation, the re-transmitted ray will have the polarizational components $b_{v,v}$ and $b_{h,v}$, represented in a vector form as:

$$\begin{bmatrix} b_{v,v} \\ b_{h,v} \end{bmatrix} = a_v = \begin{bmatrix} s_{v,v} \\ s_{h,v} \end{bmatrix} = a_h \quad (2)$$

Analogously for the second cluster corresponding to $\Pi_k(B)$, any scatterer there will reflect only the horizontally polarized component $a_h$. In vector-form,

$$\begin{bmatrix} b_{v,h} \\ b_{h,h} \end{bmatrix} = a_v = \begin{bmatrix} s_{v,h} \\ s_{h,h} \end{bmatrix} = a_h \quad (3)$$

These two clusters will each have a Gaussian Poisson intensity (not a Gaussian distribution). That is,

$$\Lambda_v(dz) = \frac{1}{2\pi\Sigma_v} \exp\left(-\frac{|z - z_{MS}|^2}{2\Sigma_v^2}\right) dz, \quad (4)$$

$$\Lambda_h(dz) = \frac{1}{2\pi\Sigma_h} \exp\left(-\frac{|z - z_{MS}|^2}{2\Sigma_h^2}\right) dz, \quad (5)$$

respectively for the first and the second clusters. These two clusters overlap each other spatially. Both their Poisson intensities are spatially co-centered at the transmitter, but may have different spatial spreads, $\Sigma_v$ and $\Sigma_h$. The two corresponding Poisson fields, $\Pi_v(dz)$ and $\Pi_h(dz)$, are modeled as statistically independent.

Hence, the overall scatterers are distributed randomly according to an heterogeneous Poisson law with a Gaussian-mixture intensity,

$$\Lambda(dz) = \frac{1 - \gamma}{2\pi\Sigma_v} \exp\left(-\frac{|z - z_{MS}|^2}{2\Sigma_v^2}\right) dz$$

$$\Lambda(dz) = \frac{\gamma}{2\pi\Sigma_h} \exp\left(-\frac{|z - z_{MS}|^2}{2\Sigma_h^2}\right) dz$$

$$(6)$$

where the model parameter $\gamma$ regulates the relative preponderance of the two clusters.\(^2\) That which is Gaussian here is the parameter $\Lambda_v(dz)$, not the spatial distribution.

The model of (6) takes on these meanings: The expected number of scatterers decreases farther from the mobile. A scatterer nearer the mobile transmitter is less likely to have an obstacle blocking its line-of-sight path. Hence, those scatterers closer to the mobile transmitter would likely have more impact on the uplink’s overall multipath profile. This is roughly equivalent to having more single-bounce scatterers closer to the mobile transmitter. Rather than modeling each scatterer’s retransmission characteristics to depend on its spatial location, it is mathematically simpler here to model all scatterers as having identical re-transmission property and as more densely populated where closer to the mobile.

\(^2\)The "geometric modeling" details here are admittedly not based on electromagnetics, but represent a mathematical construction with sufficient degrees of freedom to fit of measured data.

### III. Spatial Correlation Accounting for Polarization of the Transmitting / Receiving Antennas

#### A. Preliminary Geometric Analysis

Consider a multipath, at whatever polarization, bouncing off a scatterer located at $z = (z_x, z_y)$ and arriving at receiving-antenna #1 located at $z_{BS1} = (0, 0)$. This arriving multipath would equal the transmitted signal multiplied by the complex-valued coefficient $c_1(z) = g_1(z)e^{j\varphi(z)}$, where $g_1(z)$ represents the channel-gain. As a multipath reflects off any specific scatterer located at any particular $z$, that multipath’s initial phase $\varphi(z)$ is modeled as uniformly random over $(-\pi, \pi]$ and statistically independent of $\Pi_2(R^2)$. Any two scatterers’ initial phases are modeled as statistically independent.

Similarly for a multipath arriving at the receiving-antenna #2 located at $z_{BS2} = (d_{sp}, 0)$,

$$c_2(z) = g_2(z)e^{j(\varphi(z) - \Delta_\varphi(z))},$$

with $g_2(z)$ being the channel gain experienced by this multipath.

Referring to Figure 1,

$$s_1(z)^2 = a(z)^2 + d_1^2 + 2a(z)d_1\cos(\alpha(z) - \gamma_1(z_{MS}))$$

$$s_2(z)^2 = a(z)^2 + d_2^2 + 2a(z)d_2\cos(\alpha(z) - \gamma_2(z_{MS})),$$

where $a(z) = |z - z_{MS}|$ denotes the distance between the scatterer and the transmitter. This gives a multipath’s temporal phase-difference between receiving-antennas #1 and #2 as

$$\Delta_\varphi(z) = \frac{2\pi}{\lambda} \left[ s_1(z) - s_2(z) \right]$$

$$= \frac{2\pi}{\lambda} \left[ \sqrt{d_1^2 + 2a(z)d_1\cos(\alpha(z) - \gamma_1(z_{MS})) + a(z)^2} \right.$$  

$$- \left. \sqrt{d_2^2 + 2a(z)d_2\cos(\alpha(z) - \gamma_2(z_{MS})) + a(z)^2} \right]$$

$$\simeq \frac{2\pi}{\lambda} \left[ d_1 - d_2 + a(z) \left( \zeta_x \cos(\alpha(z) - \zeta_y \sin(\alpha(z)) \right) \right]$$

$$= \frac{2\pi}{\lambda} \left[ d_1 - d_2 + \langle \zeta, z - z_{MS} \rangle \right]$$

(7)

where $\lambda$ symbolizes the wireless signal’s carrier-wavelength, $\langle v_1, v_2 \rangle$ denotes an inner vector-product between two size-compatible vectors $v_1$ and $v_2$, $\zeta = [\zeta_x, -\zeta_y]$, $\zeta_x = d_1^2 + d_2^2 \left[ d_{sp} - (d_1 - d_2) \cos \theta \cos \beta \right]$, and $\zeta_y = -\frac{d_1^2 - d_2^2}{d_1 + d_2} (d_1 - d_2) \cos \theta \sin \beta$. The above approximation [2] holds for $d_c \gg \max\{\Sigma_v, \Sigma_h\}$ and $d_c \gg d_{sp}$. These two inequalities together require each scatterer to be sufficiently close to the transmitter relative to each scatterer’s distance from the receiver. An example field-scenario would be a base-station receiver elevated on a tower, with no prominent scatterer in the tower’s immediate vicinity.

#### B. Deriving the Closed-Form Explicit Formula

The vector-sum of the stochastic fading coefficients of all multipaths arriving at receiving-antenna #1 equals:

$$r_1 = \left[ \begin{array}{c} r_{1,v} \\ r_{1,h} \end{array} \right] = \int_{R^2} c_1(z)\text{Sali}(dz)$$

$$= \int_{R^2} g_1(z)e^{j\varphi(z)}\text{Sali}(dz)$$

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Analogously at base-station-antenna \#2, \[ E\{r_2\} = \int_{R^2} E\{e^{i\varphi(z)}\} E\{g_2(z)e^{-j\Delta\varphi(z)}SaI_v(dz)\} \]

with \[ E\{r_1\} = (1-\gamma)b_v \int_{R^2} E\{e^{i\varphi(z)}\} E\{g_1(z)\Pi_v(dz)\} \]

\[ = 0. \]

Analogously at base-station-antenna \#2, \[ E\{r_2\} = \int_{R^2} E\{e^{i\varphi(z)}\} E\{g_2(z)e^{-j\Delta\varphi(z)}SaI_v(dz)\} \]

because \( \Delta\varphi_z \) depends statistically only on \( \Pi \). If \( d_c \gg \max\{\Sigma_v, \Sigma_h\} \), (7) gives [2]:

\[ r_2 = \left[ \begin{array}{c} r_{2,v} \\ r_{2,h} \end{array} \right] = e^{-j2\pi(d_1(d_2)d_2)} \int_{R^2} g_2(z)e^{i\varphi(z)-j2\pi(z-x_{MS})\lambda}SaI_v(dz) \]

\[ = (1-\gamma)e^{-j2\pi(d_1(d_2)d_2)}b_v \]

\[ \cdot \int_{R^2} g_2(z)e^{i\varphi(z)-j2\pi(z-x_{MS})\lambda} \Pi_v(dz) \]

\[ + \gamma b_h \int_{R^2} E\{e^{i\varphi(z)}\} E\{g_1(z)\Pi_h(dz)\} \]

\[ = 0. \]

Fig. 2. The \( \rho_{vv} \left( \frac{d_{sp}}{\lambda} \right) \) formulas in (21) and the \( \rho_{bh} \left( \frac{d_{sp}}{\lambda} \right) \) formula in (22), calibrated together to the empirical data in Figure 3 of [1] with the base-station at 30 meters aboveground. \( \frac{d_{sp}}{\lambda} = 0.057339, \frac{d_{bh}}{\lambda} = 0.75787, a_v = 0.4631, s_{v,v} = 0.39693, s_{v,h} = 0.5742, \beta = 39.996^6, \gamma = 0.36398. \]

The 2 \times 2 spatio-polarizational cross-correlation matrix thus equals:

\[ R \left( \frac{d_{sp}}{\lambda} \right) = E\{r_1 r_2^*\} = \left[ \begin{array}{cc} R_{vv} \left( \frac{d_{sp}}{\lambda} \right) & R_{vh} \left( \frac{d_{sp}}{\lambda} \right) \\ R_{hv} \left( \frac{d_{sp}}{\lambda} \right) & R_{hh} \left( \frac{d_{sp}}{\lambda} \right) \end{array} \right] \]

\[ = (1-\gamma)^2b_vb_h^H \int_{R^2} g_1(z)g_2(z)e^{j\Delta\varphi(z)} \Lambda_v(dz) \]

\[ + \gamma^2 b_h b_h^H \int_{R^2} g_1(z)g_2(z)e^{j\Delta\varphi(z)} \Lambda_h(dz) \]

\[ = (1-\gamma)^2b_vb_h^H \int_{R^2} g_1(z)g_2(z)e^{j2\pi(d_1(d_2))} \Lambda_v(dz) \]

\[ + \gamma^2 b_h b_h^H \int_{R^2} g_1(z)g_2(z)e^{j2\pi(d_1(d_2))} \Lambda_h(dz) \]
The entries in the $2 \times 2$ matrix of (16) are:

$$R_{vv} \left( \frac{d_{sp}}{\lambda} \right) \approx (1 - \gamma)^2 (s_{v,v} a_v)^2 e^{-2 \left( \frac{\pi d_{sp}}{\lambda} \frac{\Sigma z_2}{\Sigma z_1} \sin \beta \right)^2}$$

$$+ \gamma^2 (s_{v,h} a_h)^2 e^{-2 \left( \frac{\pi d_{sp}}{\lambda} \frac{\Sigma z_1}{\Sigma z_2} \sin \beta \right)^2}$$

(17)

$$R_{hh} \left( \frac{d_{sp}}{\lambda} \right) \approx (1 - \gamma)^2 (s_{h,v} a_v)^2 e^{-2 \left( \frac{\pi d_{sp}}{\lambda} \frac{\Sigma z_2}{\Sigma z_1} \sin \beta \right)^2}$$

$$+ \gamma^2 (s_{h,h} a_h)^2 e^{-2 \left( \frac{\pi d_{sp}}{\lambda} \frac{\Sigma z_1}{\Sigma z_2} \sin \beta \right)^2}$$

(18)

$$R_{hv} \left( \frac{d_{sp}}{\lambda} \right) = \left| R_{vh} \left( \frac{d_{sp}}{\lambda} \right) \right| \approx (1 - \gamma)^2 (s_{v,h} a_h)^2 e^{-2 \left( \frac{\pi d_{sp}}{\lambda} \frac{\Sigma z_2}{\Sigma z_1} \sin \beta \right)^2}$$

$$+ \gamma^2 (s_{h,v} a_v)^2 e^{-2 \left( \frac{\pi d_{sp}}{\lambda} \frac{\Sigma z_1}{\Sigma z_2} \sin \beta \right)^2}$$

(19)

The corresponding spatial correlation coefficient functions may be found by normalizing the diagonal elements of $\mathbf{R} (0)$ in (16) to unity magnitude. That is,

$$\rho \left( \frac{d_{sp}}{\lambda} \right) = \begin{bmatrix} \rho_{vv} \left( \frac{d_{sp}}{\lambda} \right) & \rho_{vh} \left( \frac{d_{sp}}{\lambda} \right) \\ \rho_{hv} \left( \frac{d_{sp}}{\lambda} \right) & \rho_{hh} \left( \frac{d_{sp}}{\lambda} \right) \end{bmatrix}$$

$$= \begin{bmatrix} |R_{vv} (0)| & 0 \\ 0 & |R_{hh} (0)| \end{bmatrix}^{-1/2} \begin{bmatrix} |R_{vv} \left( \frac{d_{sp}}{\lambda} \right) | & 0 \\ 0 & |R_{hh} \left( \frac{d_{sp}}{\lambda} \right) | \end{bmatrix}^{-1/2}$$

The above $2 \times 2$ matrix, $\rho \left( \frac{d_{sp}}{\lambda} \right)$, has the elements shown in equation (21) to (24).

In the above, $\rho_{vv} \left( \frac{d_{sp}}{\lambda} \right)$ refers to the spatial correlation coefficient between two vertically polarized antennas (apart by a horizontal distance of $\frac{d_{sp}}{\lambda}$) at the receiver. Similarly, $\rho_{hh} \left( \frac{d_{sp}}{\lambda} \right)$ refers to the spatial correlation coefficient between two horizontally polarized antennas (apart by a horizontal distance of $\frac{d_{sp}}{\lambda}$) at the receiver. Lastly, $\rho_{vh} \left( \frac{d_{sp}}{\lambda} \right)$ and $\rho_{hv} \left( \frac{d_{sp}}{\lambda} \right)$ each refers to the spatial correlation coefficient between one vertically polarized antenna and one horizontally polarized antenna (apart by a horizontal distance of $\frac{d_{sp}}{\lambda}$) at the receiver.

This model’s independent parameters are

(i) $\frac{\Sigma z_1}{\lambda}$ and $\frac{\Sigma z_2}{\lambda}$, which refer to the channel’s wavelength-normalized scattering environment;

(ii) $\beta$, which geometrically relates the mobile station and the base-station; and

(iii) $\frac{d_{sp}}{\lambda}$, which represents a normalized design parameter for the base-station receiving antenna-array’s aperture.

The closed-form formulas in (21) and (24), explicitly expressed in terms of the geometric-model’s independent parameters, can thus serve as a first-step design-formulas in the development of MIMO or smart-antennas systems.

Figures 3 and 2 show the above derived $|\rho_{vv} \left( \frac{d_{sp}}{\lambda} \right)|$ of (21) and $|\rho_{hh} \left( \frac{d_{sp}}{\lambda} \right)|$ of (22), calibrated (by least-squares fit) to the empirical data shown in Figure 3 of [1]. These empirical measurements were for signals transmitted at street level and
collected by a base-station receiver at 100 or 30 meters high. Figures 3 and 2 show the proposed model’s efficacy to model real-world spatial correlation as function of the transmitting and the receiving antennas’ polarizations.

C. The Derived Formulas’ General Qualities

Besides Figures 3 and 2’s validation of the derived formulas in (21) and (24) by empirical data, (21) and (24) exhibit these following intuitively appealing qualitative trends. \( |\rho \left( \frac{d_{sp}}{\lambda}, \beta \right) | \) decreases quasi-exponentially with

1) increasing \( \sin^2 \beta \), as the cluster impinges from a spatial direction more perpendicular to the antenna-array’s axis.
2) increasing \( \left( \frac{d_{sp}}{\lambda} \right) \) as the base-station’s antennas are spaced farther apart, except for \( \beta \approx 0 \) when the mobile aligns along the antenna-array axis.
3) increasing \( \left( \frac{\Sigma}{d_{me}} \right)^2 \) as the first cluster’s azimuth-angular spread appears larger to the base-station, except for \( \beta \approx 0 \) when the mobile aligns along the antenna-array axis.

Similar considerations apply for \( \left( \frac{\Sigma}{d_{me}} \right)^2 \) for the second cluster.

IV. Conclusion

This paper is first in the open literature to account for the transmitting and receiving antennas’ polarizations, in geometric-model-based derivation of truly closed-form expressions for the spatial correlation coefficient across a receiving antenna-array. The derived formulas are explicitly in terms of (1) the inter-antenna spacing between two receiving-antennas, and (2) the two receiving-antenna’s linear polarizations. These formulas are simple in mathematical form and fit some empirical data well.

REFERENCES