S-separability of the far-field beam-pattern due to an arbitrary excitation-function over an elliptical-rim aperture

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Abstract: This study is the first in the phase-mode analysis literature to analytically characterise the far-field azimuth-elevation (φ-θ) beam-pattern due to any arbitrary excitation-function on an elliptical-rim aperture. The key findings include: such an arbitrary excitation-function’s far-field beam-pattern is not always φ-θ separable. Nonetheless, this beam-pattern is S-separable in the sense that it is a sum of φ-θ separable terms. Moreover, this sum may be approximated (with arbitrarily small truncation-error) as a finite-term summation. A ceiling of this truncation-error is derived.

1 Introduction

1.1 Significance of the elliptical-rim aperture

Elliptical-rim apertures have been physically implemented in [1] and [2], and used for interference rejection in [3]. Co-array synthesis has been investigated in [4] for an elliptical-rim boundary-aperture.

An elliptical-rim aperture combines these two advantages:

1. An elliptical aperture, more than a circular aperture, can selectively enhance the resolution of specific azimuth sectors, namely those pointing roughly along the ellipse’s major axis.

2. For a sensor array, its imaging resolution critically depends on the array’s geometric size; and a boundary array would span the largest physical aperture for a given number of array elements and a given inter-element spacing.

Several earlier papers have investigated the beam pattern of a spatially continuous elliptical-rim aperture: [5] assumes a uniform excitation-function. In [6–9], the authors investigate a few specific excitation-functions. Ji et al. [10] consider only elliptically symmetric point spread functions. All these results are inapplicable to a general excitation-function.

In [11] and [12], algorithms are proposed to identify excitation functions for a spatially discrete elliptical-rim aperture to satisfy certain specified constraints in the beam pattern; however, the results therein are inapplicable to an arbitrary excitation-function.

In [13], an ‘equivalence theory’ relates the elliptical-rim’s far-field beam-pattern to its counterpart beam-pattern on a circular-ring aperture. However, no explicit expression is therein presented for the beam pattern of a general arbitrary excitation function. In [3, 14, 15], the ‘equivalence theory’ is applied for numerical computations, but they offer no analytic expression of the beam pattern arisen from an arbitrary excitation-function.

1.2 Phase-mode analysis

Phase-mode excitation analysis may be applied to any aperture, whether the aperture is spatially continuous or discrete. An array of spatially displaced sensors may be conceptualised as spatial samples of a continuous virtual array-aperture, forming a spatially discrete aperture. The aperture may be one-, two- or three-dimensional in space.
Phase-mode excitation analysis involves the following key concepts:

1. When the aperture operates in the transmission mode, ‘excitation’ refers to transmission from various parts of the aperture. When the aperture operates in the reception mode, ‘excitation’ refers to the summing weight assigned to data collected by the aperture’s different parts. (Recall that beamforming is fundamentally a weighted sum of various sensors’ data.)

2. For a spatially continuous aperture, the excitation function is mathematically defined over the aperture’s entire geometric expanse, and may be assumed to have continuous derivatives up to any order required. For a spatially discrete aperture (i.e. an array of spatially displaced sensors), the excitation function is physically actualised only at the individual sensors’ discrete locations. These sensors’ collective excitation may be viewed as spatial samples of a virtual spatially continuous excitation function defined over the entire virtual continuous aperture.

3. A smooth (whether actual or virtual) excitation function may be mathematically expressed in terms of a set of orthonormal basis functions, which is characteristic of the geometry of the array’s virtual aperture. This orthonormal basis depends only on the aperture’s spatial geometry; however, each aperture geometry may have more than one such set of orthonormal basis functions. For a discrete aperture, this orthonormal basis is independent of the spatial sampling grid that defines the sensors’ actual locations on the virtual continuous aperture.

4. Each aforementioned basis function’s individual contribution to the array’s overall beam-pattern constitutes one operational mode of the sensor array. The aperture’s overall far-field beam pattern sums each mode’s individual beam pattern.

5. Phase-mode excitation analysis is predicated on this assumption: Any of the aperture’s physically realisable beam patterns can be ‘adequately’ approximated as arisen from only a finite subset of all possible operational modes. This assumption needs to be proven case-by-case for each aperture geometry under consideration. This paper is first in the open literature to prove this assumption for the elliptical-rim aperture for an arbitrary excitation-function in the open literature to prove this assumption case-by-case for each assumption needs to be proven. This assumption: Any of the aperture’s physically realisable beam pattern to be \( \Sigma \)-separable. However, this \( \Sigma \)-separability generally would involve a double infinite-term summation. Section 4 presents various truncation analyses to reduce the double infinite-term summation to finite terms. Section 5 illustrates this \( \Sigma \)-separability and the truncation analyses with a numerical example. Section 6 concludes the paper. The appendices give the detailed proofs.

2 Mathematical basics

2.1 Geometrical basics on an elliptical rim

An elliptic rim \( A \), with half-axis lengths \( a, b > 0 \), may be represented in the Cartesian coordinates as \( A \equiv \{(x, y) \in \mathbb{R}^2: x^2/a^2 + y^2/b^2 = 1\} \). Consider a polar-coordinates system \((r, \alpha)\) with the polar-axis being the positive \(x\)-axis and the pole at the origin. Please see Fig. 1. The ellipse may be expressed as

\[
\begin{align*}
  r &= r(\alpha) \equiv \left(\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}\right)^{-1/2}, \quad \alpha \in [0, 2\pi] \\
  \epsilon &\equiv \max\{a, b\}.
\end{align*}
\]

Let \( \epsilon \) denote the half-length of the major axis of the ellipse, that is, \( \epsilon \equiv \max\{a, b\} \). For a point \((r, \alpha)\) on the elliptical rim, the infinitesimal arc-length element \( s(\alpha) \, d\alpha \) corresponds to an increment \( d\alpha \) in \( \alpha \), where

\[
  s(\alpha) \equiv r(\alpha) \left[1 + r^2(\alpha) \sin^2(\alpha) \cos^2(\alpha) \left(b^2 - a^2\right)^2\right]^{1/2}
\]

This idealises the elliptical rim as an infinitely narrow wire. Consideration of wire thickness and width would require generalisation of the present two-dimensional formulation to a three-dimensional surface or volume. A distinct (and a much more difficult) set of mathematical derivations is needed to analyse this three-dimensional case.

Consider an incident source, emitting at wavelength \( \lambda \) from an azimuth-angle \( \phi \) (measured from the positive \(x\)-axis) and an elevation angle \( \theta \) (measured from the positive \(z\)-axis) in the far-field. As this incident source impinges upon a sensor at \((s, y) = (r(\alpha) \cos \alpha, r(\alpha) \sin \alpha)\) on the elliptical rim, there will incur a relative phase of

\[
\frac{2\pi}{\lambda} \sin \theta (r(\alpha)(\cos \phi \cos \alpha + \sin \phi \sin \alpha))
\]

with reference to the corresponding phase for a hypothetical sensor at the Cartesian origin.

1.3 This paper’s outline

Section 2.1 reviews some geometrical basics on an elliptical rim needed for subsequent development. Section 2.2 reviews basic facts and notations associated with an orthonormal basis, needed for phase-mode analysis. Section 3 (i) derives the far-field beam-pattern due to an arbitrary excitation-function over any elliptical-rim aperture, (ii) proves that this beam pattern is generally not \( \phi-\theta \) separable, (iii) but proves this beam pattern to be \( \Sigma \)-separable. However, this \( \Sigma \)-separability generally would involve a double infinite-term summation.
Over the entire continuous elliptical-rim virtual aperture \( \mathcal{A} \), the excitation function \( f(\alpha) \) would produce a far-field beam-pattern

\[
B(\phi, \theta) \overset{\text{def}}{=} \int_{0}^{2\pi} e^{i\phi r(\alpha) \cos(\phi - \alpha)} f(\alpha) s(\alpha) \, d\alpha \quad \text{(4)}
\]

**2.2 Basic facts and notations associated with an orthonormal basis**

Let \( L^2([0, 2\pi], s(\alpha) \, d\alpha) \) be the classical Hilbert space of all square-integrable functions \( f(\alpha) \) defined on the elliptical rim. That is, \( \int_{0}^{2\pi} |f(\alpha)|^2 s(\alpha) \, d\alpha < \infty \). Incidentally, an additional ‘normality’ assumption of \( \int_{0}^{2\pi} |f(\alpha)|^2 s(\alpha) \, d\alpha = 1 \) would not further simplify the subsequent analysis. This space includes all space-time-continuous excitation-functions that are physically realisable. Let the set \( \{\tau_n(\alpha), -\infty < n < \infty\} \) consist of functions forming an ordered orthonormal basis for the space \( L^2([0, 2\pi], s(\alpha) \, d\alpha) \). Then

\[
\|f(\alpha)\| \overset{\text{def}}{=} \sqrt{\int_{0}^{2\pi} |f(\alpha)|^2 s(\alpha) \, d\alpha} \quad \text{(the norm of } f(\alpha) \text{)} \quad \text{(5)}
\]

\[
f_n \overset{\text{def}}{=} \int_{0}^{2\pi} f(\alpha)[\tau_n(\alpha)]^* s(\alpha) \, d\alpha \quad \text{(6)}
\]

where \( \tau_n(\alpha) \) could refer to the \( n \)th mode of excitation, \( f_n \) symbolises the \( n \)th mode’s coefficient of \( f(\alpha) \) and the superscript * denotes complex conjugation.

From classical Hilbert space theory

\[
f(\alpha) = \sum_{-\infty < n < \infty} f_n \tau_n(\alpha) \quad \text{[in } L^2([0, 2\pi], s(\alpha) \, d\alpha)] \quad \text{(7)}
\]

\[
\|f(\alpha)\|^2 = \sum_{-\infty < n < \infty} |f_n|^2 \quad \text{(Parseval’s identity)} \quad \text{(8)}
\]

**3 \( \Sigma \)-separability of an arbitrary excitation-function’s beam-pattern**

Appendix 1 proves the following proposition:

**Proposition 1:** The beam-pattern \( B(\phi, \theta) \) may be expressed as a sum of mode-specific beam-patterns as follows

\[
B(\phi, \theta) = \sum_{-\infty < n < \infty} f_n \int_{0}^{2\pi} e^{i\phi r(\alpha) \cos(\phi - \alpha)} \tau_n(\alpha) s(\alpha) \, d\alpha = B_n(\phi, \theta) \quad \text{(9)}
\]

where \( B_n(\phi, \theta) \) denotes the far-field beam-pattern due to only the \( n \)th excitation-mode.

The next proposition implies the non-existence of any basis over the elliptical rim whereby \( B(\phi, \theta) \) is \( \phi-\theta \) separable for all excitation functions on the elliptical rim.

**Proposition 2:** The bivariate function \( B(\phi, \theta) \) may not be separable into a product of some function \( h(\phi) \) which is independent of \( \theta \), with some function \( g(\theta) \) which is independent of \( \phi \).

Appendix 2 proves the above proposition. [For an arbitrary excitation-function on the elliptical-rim aperture, the earlier mentioned ‘equivalence theory’ [13] would also produce a generally \( \phi-\theta \) inseparable far-field beam-pattern, even as the corresponding beam pattern is likewise \( \phi-\theta \) inseparable for the circular-ring aperture.]
Nonetheless, $B(\phi, \theta)$ is ‘$\Sigma$-separable’, in the sense that there exist functions $G_k(\theta)$ (that are independent of $\phi$) and some other functions $H_k(\phi)$ (that are independent of $\theta$) such that

$$B(\phi, \theta) = \sum_{k=0}^{\infty} \tilde{G}_k(\theta)\tilde{H}_k(\phi)$$

**Theorem 1:** Let

$$\tilde{G}_k(\theta) = G_k(\theta) \overset{\text{def}}{=} \int_{-L/2}^{L/2} (2\pi)^t \sin^t \theta$$

$$\tilde{H}_k(\phi) = H_k(\phi) \overset{\text{def}}{=} \sum_{-\infty<\nu<\infty} f_k F_{k,\nu}(\phi)$$

$$F_{k,\nu}(\phi) \overset{\text{def}}{=} \int_{0}^{2\pi} \frac{r^k(\alpha)}{\lambda^t} \cos^t(\phi - \alpha)c_{\nu}(\alpha)\tau_{\nu}(\alpha)\, d\alpha$$

Then

$$B(\phi, \theta) = \sum_{k=0}^{\infty} \tilde{G}_k(\theta)\tilde{H}_k(\phi)$$

Appendix 3 proves the above theorem. Note that $G_k(\theta)$ is $2\pi$-periodic in $\theta$ on the real-number line $\mathbb{R}$, whereas $H_k(\phi)$ and $F_{k,\nu}(\phi)$ are $2\pi$-periodic in $\phi$ on $\mathbb{R}$.

## 4 Truncation of the infinite-term double summation in (13)

A double infinite summation exists implicitly in (13), after substituting in (11). This infinite-term double summation may be approximated (with arbitrary closeness) by a finite-term double summation, in two steps as shown in Sections 4.1 and 4.2. This second truncation-step of Section 4.2 could be improved, in a certain sense (to be later defined), by the alternative truncation stated in Section 4.3. Section 4.4 gives a quantitative ceiling of the error in $B(\phi, \theta)$ resulting from the truncations in Sections 4.1–4.3.

The above-mentioned truncation analysis depends on the convergence rate of the series $\sum_{-\infty<\nu<\infty} |f_k|^2$. Appendix 8 will present yet another truncation-error analysis that makes no reference to the aforementioned convergence rate, but that requires additional pre-conditions on the excitation function $f(\alpha)$ regarding ‘smoothness’ and additional pre-conditions on the orthonormal basis.

### 4.1 Truncation of the infinite sum in (13)

Appendix 4 will prove the following theorem, which truncates infinite-term summation stated in (13) to a finite number of terms.

**Theorem 2:** Let any $\epsilon > 0$ and for any excitation-function $f(\alpha)$ that is not identically zero (i.e. for $\|f(\alpha)\| > 0$), define a non-negative number $K_\epsilon$ as follows

$$K_\epsilon \overset{\text{def}}{=} \max \left\{ \epsilon, \frac{1}{\ln 2} \left[ \ln \sqrt{\|f(\alpha)\|/(4\pi\lambda)^t} \right] \right\}$$

$$s \overset{\text{def}}{=} \max \left\{ \frac{4\pi\lambda}{\epsilon} \right\}$$

where $\epsilon \overset{\text{def}}{=} \int_{\lambda}^{\lambda} |s(\alpha)|\, d\alpha$ denotes the total arc-length of the elliptical rim, and $s(\alpha)$ symbolises the half-length of the major axis of the ellipse. If $f(\alpha)$ is identically zero (i.e. $\|f(\alpha)\| = 0$), set $K_\epsilon = 0$

Then, for every integer $K \geq K_\epsilon$

$$\sum_{k=K+1}^{\infty} |G_k(\theta)H_k(\phi)| < \epsilon$$

for all $(\phi, \theta) \in \mathbb{R}^2$. Consequently

$$|B(\phi, \theta) - \sum_{k=0}^{K} G_k(\theta)H_k(\phi)| < \epsilon$$

for all $(\phi, \theta) \in \mathbb{R}^2$, and the convergence in (13) is uniform over $[\{\phi, \theta\} | \phi, \theta \in \mathbb{R}]$.

The definition in (14) is a convenient choice to account for the number $L$ in (29) of the Appendix. Note that $K_\epsilon$ depends only on $\epsilon$, $\|f(\alpha)\|$, $c/\lambda$, and $C$; however, $K_\epsilon$ is independent of $\theta$, $\phi$ and the choice of the orthonormal basis $(\tau_{\nu}(\alpha))_{-\infty<\nu<\infty}$.

### 4.2 Truncation of the infinite sum in (11)

The infinite summation in (11) may be truncated with arbitrarily small error, by Lemma 1 and Theorem 3 below. (Note that this infinite summation shows up in (16) and (17).) Appendix 5 gives the proof of this lemma and theorem.

**Lemma 1:** For all non-negative integer $k$, $H_k(\phi) \overset{\text{def}}{=} \sum_{-\infty<\nu<\infty} f_k F_{k,\nu}(\phi)$ may be replaced in (16) and (17) by any ‘sufficiently long’ finite partial sum

$$H_k^{(N)}(\phi) \overset{\text{def}}{=} \sum_{|\nu| \leq N} f_k F_{k,\nu}(\phi)$$

where $N$ is independent of $k$. In fact, it suffices that $N \geq Q$, ...
with $Q$ being the smallest non-negative integer such that
\[
\left\| f(\alpha) \right\|^2 - \sum_{|n| \leq Q} |f_n|^2 \leq \frac{e^{-4m/\lambda^2}}{\ell}
\] (19)

\[
\delta \equiv \epsilon - \left\| f(\alpha) \right\| \sqrt{\ell} \sum_{k=0}^{\infty} \frac{1}{K^k} \left[ \frac{2\pi \sin k\ell}{\lambda} \right] (20)
\]

Appendix 5 also proves that $\delta \in (0, \epsilon]$. Note that $Q$ depends only on $\epsilon$, $\left\| f(\alpha) \right\|$, $c/\lambda$, $\ell$ and the rate of convergence in $\sum_{|n| < n \leq \infty} |f_n|^2$ (hence indirectly dependent on the choice of the orthonormal basis), but $Q$ does not depend on $(\phi, \theta) \in \mathbb{R}^2$.

**Theorem 3**: Let $K_e$ be given by (14) and (15). Let $Q$ be given as in Lemma 1. Then, for every integer $K \geq K_e$ and every integer $N \geq Q$

\[
B(\phi, \theta) \equiv \sum_{n=-N}^{N} f_n \sum_{k=0}^{K} G_k(\theta) F_{k,n}(\phi) < \epsilon
\] (21)

for any arbitrarily set $\epsilon > 0$.

The double infinite-term summation in (17) may thus be truncated (with arbitrarily small error) to a double finite-term summation. This means that excitation modes outside of $\{ -N, 1 - N, \ldots, N - 1, N \}$ may be overlooked, and that each excitation mode in $\{ -N, 1 - N, \ldots, N - 1, N \}$ may each be approximated as a finite sum.

**4.3 Alternative truncation of the infinite sum in (11)**

It may be difficult to evaluate $Q$, because that would require evaluating $\delta$ and thus $\sum_{|n| \leq Q} (1/|k|)(2\pi/\lambda)^k$. To bypass $Q$ altogether, Theorem 4 can help. Appendix 6 proves Theorem 4.

**Theorem 4**: For an arbitrarily given $\epsilon > 0$, let $K_e$ be given by (14) and (15) as before, and according to (8), let $N_e$ be the first non-negative integer such that

\[
\left\| f(\alpha) \right\|^2 - \sum_{|n| \leq N_e} |f_n|^2 \leq \frac{\epsilon^2}{\ell}
\] (22)

Then for every integer $K \geq K_e$ and every integer $N \geq N_e$

\[
B(\phi, \theta) \equiv \sum_{n=-N}^{N} f_n \sum_{k=0}^{K} G_k(\theta) F_{k,n}(\phi) < 2\epsilon
\] (23)

for all $(\phi, \theta) \in \mathbb{R}^2$. That is, (21) remains valid with a larger truncation error bound.

By (39), $0 < \delta \leq \epsilon$; thus, $N_e \leq Q$ by comparing the definitions in (19) and (22). However, $N_e$ gives rise to a truncation error bound exactly twice that of $\delta Q$, by comparing (21) against (23). [The mode-specific beam-pattern $B_n(\phi, \theta)$ is likewise approximated by a finite sum $\sum_{k=0}^{K} G_k(\theta) F_{k,n}(\phi)$. This is perhaps the best available alternative to the $\phi-\theta$ separability in $B_n(\phi, \theta)$, which might be unattainable.]

The positive integer $N_e$ depends (only) on $\epsilon$, $\ell$, and the convergence rate of the series $\sum_{|n| = 0}^{\infty} |f_n|^2$. Therefore, $N_e$ depends indirectly on the choice of the orthonormal basis. [If $\theta$ is fixed, $K_e$ may be replaced in everything above (except the uniform convergence) by the following (possibly smaller) non-negative number:

\[
K_{e,0} \equiv \max \left\{ t, \frac{1}{ln 2} \left[ \ln \sqrt{\ell} \left\| f(\alpha) \right\| (4\pi c \sin \theta) \frac{\epsilon}{\ell N e} \right] + \frac{4\pi c \sin \theta}{\lambda} \right\}
\]

(24)

The non-negative number $K_{e,0}$ (like $K_e$) is independent of the orthonormal basis used. Specifically, (17), (16), (35), (21) and (23) clearly remain valid if all $K_e$ therein is replaced by $K_{e,0}$]

**4.4 Quantitative ceiling of the truncation error of $B(\phi, \theta)$**

The following theorem upper-bounds the overall truncation-error in $B(\phi, \theta)$, caused by the truncations in Sections 4.1–4.3.

**Theorem 5**: For every pair of non-negative integers $K$ and $N$

\[
\left\| f(\alpha) \right\|^2 - \sum_{k=0}^{K} G_k(\theta) H_k^{(N)}(\phi) < \frac{\epsilon^2}{\ell} \left( \sum_{|n| > N} |f_n|^2 \right)^{1/2} + \left\| f(\alpha) \right\| \sqrt{\ell} \sum_{k=K+1}^{\infty} \frac{1}{K^k} \left[ \frac{2\pi \sin k\ell}{\lambda} \right]^k
\]

(25)

Appendix 7 proves the above theorem and derives ceilings of $G_k(\theta), F_{k,n}(\phi), H_k(\phi)$ in (43)–(45), respectively.

**5 Illustrative example**

Herein presented is a numerical example to illustrate the analysis in Section 4, which finds suitable cut-off points $(K_e, Q, N_e, N_e)$ for an arbitrarily set truncation-error tolerance-level (i.e. $\epsilon$) in $B(\phi, \theta)$.

Let $a = \sqrt{0.5}$ and $b = 2.5$, giving an elliptical rim of

\[
r(\alpha) = (2 \cos^2 \alpha + 0.16 \sin^2 \alpha)^{-1/2}, \quad \alpha \in [0, 2\pi]
\] (26)

Consider a uniform excitation function on the above elliptical rim, that is, $f(\alpha) = 1$ on the whole elliptical rim. Furthermore, let $\lambda = 0.24$. 

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**References**


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Appendix 9 proves that the $\Psi$ below constitutes an orthonormal basis for the Hilbert space $L^2([0, 2\pi], s(\alpha)\,d\alpha)$

$$\Psi = \{\tau_n(\alpha), -\infty < n < \infty\} \overset{\text{def}}{=} \{\psi_n(\alpha) = e^{in\alpha}/\sqrt{2\pi}\}$$

Then (see equation at the bottom of the page).

For $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$ (see equation at the bottom of the page)

For every odd integer $n$, because $\sqrt{\pi/2} f_n = \int_0^n \cos(n\alpha)\,d\alpha = \int_{\pi/2}^n \cos(n\alpha)\,d\alpha = \int_{\pi/2}^n \cos(n\beta)\,d\beta$ (where $\beta = \pi - \alpha$), we obtain $f_n = F_{0,n}(\phi) = 0$. Similarly, $F_{1,n}(\phi) = 0$ for all even integers $n$, and $F_{2,n}(\phi) = 0$ for all odd integers $n$.

Numerical computation gives

$$f_0 = F_{0,0}(\phi) \approx 3.15407$$
$$f_{\pm 2} = F_{0,\pm 2}(\phi) \approx -0.65496$$
$$f_{\pm 4} = F_{0,\pm 4}(\phi) \approx 0.144008$$
$$f_{\pm 6} = F_{0,\pm 6}(\phi) \approx 0.01257358$$
$$f_{\pm 8} = F_{0,\pm 8}(\phi) \approx -0.0573932$$
$$f_{\pm 10} = F_{0,\pm 10}(\phi) \approx 0.0631726$$
$$F_{1,\pm 1}(\phi) \approx 4.6946 \cos(\phi) \pm j13.2505 \sin(\phi)$$
$$F_{1,\pm 3}(\phi) \approx -2.38312 \cos(\phi) \mp j6.17726 \sin(\phi)$$
$$F_{1,\pm 5}(\phi) \approx 1.17488 \cos(\phi) \pm j2.61476 \sin(\phi)$$
$$F_{2,0}(\phi) \approx 60.5779 - 41.3533 \cos(2\phi)$$

$$c \overset{\text{def}}{=} \max[|a|, |b|] = 2.5$$

$$s(\alpha) = r(\alpha) \left(1 + r^4(\alpha) \sin^2(\alpha) \cos^2(\alpha) \right) \left(b^{-2} - a^{-2}\right)^{1/2}$$

$$\approx \sqrt{2 \cos^2(\alpha) + 0.16 \sin^2(\alpha) - 3 \cos^2(\alpha) \sin^2(\alpha)}$$

$$\approx \sqrt{20.0128 + 1.9872 \cos(2\alpha)}$$

$$\|f(\alpha)\| = \ell = \int_0^{2\pi} s(\alpha)\,d\alpha \approx 10.8794$$

$$|B(\phi, \theta)| = \left|\int_0^{2\pi} e^{j\phi r(\alpha)} f(\alpha) s(\alpha)\,d\alpha\right|$$

(by periodicity and symmetry of the integrand)

$$G_0(\theta) = \int_0^{2\pi} (2\pi)^2 \sin^4(\theta)$$

$$F_{e,\phi}(\phi) = \int_0^{2\pi} r^{-1}(\alpha) \cos^2(\phi - \alpha) \psi_n(\alpha) s(\alpha)\,d\alpha$$

$$H^{(0)}_n(\phi) = \sum_{|n| \leq N} f_n F_{e,\phi}(\phi)$$

Let $e = 1$. Then, $K_1 = 187.45$ due to (14), and $N_1 = 2$ as $N_1$ is the smallest non-negative integer such that $10.8794 - \sum_{|n| \leq N_1} |f_n|^2 < e^2 / \ell \approx 0.0919$, according to (22). By (23)

$$B(\phi, \theta) - \sum_{n=-2}^{2} f_n \sum_{k=0}^{189} G_k(\theta) F_{k,n}(\phi) \left| < 2 = 2e \right.$$  

If $e = 0.5$ instead of unity as above, then $N_{0.5} = 10$, $K_{0.5} = 188.45$ and

$$B(\phi, \theta) - \sum_{n=-10}^{10} f_n \sum_{k=0}^{189} G_k(\theta) F_{k,n}(\phi) \left| < 1 = 2e \right.$$  

6 Conclusion

This analysis is first in the phase-mode literature to characterise the far-field beam pattern arisen from an arbitrary excitation-function over an elliptical-rim aperture. $\Sigma$-separability is herein defined and proved to hold for this beam pattern, regardless of the orthonormal basis chosen. The beam pattern may be approximated, with arbitrary precision, as a finite-term double summation. A ceiling of this truncation-error is may be approximated, with arbitrary precision, as a finite-term double summation. A ceiling of this truncation-error is

All the abovementioned results actually hold for all sectionally smooth closed plane curves and not just for ellipses, with similar proofs [A sectionally smooth closed plane curve $C$ is here defined as a plane curve given, in polar coordinates, by

$$r = r(\alpha), \quad \alpha \in [0, 2\pi]$$

where the function $r(\alpha)$ is continuous on $[0, 2\pi]$, $r(0) = r(2\pi)$, the derivative $r'(\alpha)$ is well-defined and continuous on $[0, 2\pi]$ except possibly at a few points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $[0, 2\pi]$, $n \in \mathbb{N}$, and the right-hand limits $\lim_{\alpha \to \alpha_i^+} r'(\alpha)$ and the left-hand limits $\lim_{\alpha \to \alpha_i^-} r'(\alpha)$, $i = 1, 2, \ldots, n$ (one-sided limits only for the end points 0, $2\pi$), exist as finite real numbers.]  

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8 References


9 Appendix 1: proof of Proposition 1

Note that \( \{\tau_n(\alpha)\}_{n \in \mathbb{N}} \) is an orthonormal basis for the space \( L^2([0, 2\pi], \alpha(\alpha) d\alpha) \) and \( f(\alpha) \in L^2([0, 2\pi], \alpha(\alpha) d\alpha) \). Classical Hilbert space theory gives

\[
  f(\alpha) = \sum_{-\infty < n < +\infty} f_n \tau_n(\alpha) \quad \text{[in } L^2([0, 2\pi], \alpha(\alpha) d\alpha)]
\]

Hence

\[
  B(\phi, \theta) \overset{\text{def}}{=} \int_0^{2\pi} e^{i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)} f(\alpha) \alpha(\alpha) d\alpha = (f', \alpha), e^{-i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)}
\]

\[
  = \sum_{-\infty < n < +\infty} f_n \tau_n(\alpha), e^{-i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)}
\]

\[
  = \sum_{-\infty < n < +\infty} f_n \int_0^{2\pi} e^{i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)} \tau_n(\alpha) \alpha(\alpha) d\alpha
\]

which yields (9). \( \square \)

10 Appendix 2: proof of Proposition 2

\( B(\phi, \theta) \) of (4) would be separable in reference to \( \phi \) and \( \theta \), only if

\[
  B(\phi, \theta) \frac{\partial^2 B(\phi, \theta)}{\partial \phi \partial \theta} - \frac{\partial B(\phi, \theta)}{\partial \phi} \frac{\partial B(\phi, \theta)}{\partial \theta} = 0 \quad (27)
\]

To see the validity of this necessary condition, derive the partial derivatives of \( B(\phi, \theta) = g(\theta)h(\phi) \) and substitute them into (27) [incidentally, this condition is also sufficient, though sufficiency is irrelevant here.]

This necessary requirement is violated in the following counter-example. Consider an excitation-function \( f_2(\alpha) = 0 \) on \([\pi/2, 3\pi/2]\) and \( f_2(\alpha) > 0 \) on the remaining parts of \([0, 2\pi]\). Then, \( \int_0^{2\pi} f_2(\alpha) r(\alpha) d\alpha > 0 \) and \( \int_0^{2\pi} (\cos \alpha) f_2(\alpha) r(\alpha) d\alpha > 0 \), because each integrand is non-negative in the integration interval and positive in certain subintervals. Therefore

\[
  \frac{\partial B}{\partial \phi} \bigg|_{\phi = \pi/2, \theta = 0} = \int_0^{2\pi} \frac{\partial}{\partial \phi} \left[ e^{i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)} \right] |_{\phi = \pi/2, \theta = 0} f_2(\alpha) \alpha(\alpha) d\alpha = 0
\]

\[
  \frac{\partial^2 B}{\partial \phi \partial \theta} \bigg|_{\phi = \pi/2, \theta = 0} = \int_0^{2\pi} \frac{\partial^2}{\partial \phi \partial \theta} \left[ e^{i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)} \right] |_{\phi = \pi/2, \theta = 0} f_2(\alpha) \alpha(\alpha) d\alpha
\]

\[
  = -\frac{2\pi}{\lambda} \int_0^{2\pi} (\cos \alpha) f_2(\alpha) r(\alpha) \alpha(\alpha) d\alpha \neq 0
\]

\[
  B(\phi, 0) = \int_0^{2\pi} f_2(\alpha) \alpha(\alpha) d\alpha \neq 0
\]

Hence, (27) fails at \((\pi/2, 0)\). Indeed, by continuity, (27) fails in neighbourhood of \((\pi/2, 0)\).

The beam-pattern \( B(\phi, \theta) \) is thus inseparable for \( f_2(\alpha) \), in particular. \( \square \)

11 Appendix 3: proof of Theorem 1

In the trivial case of \( \|f(\alpha)\| = 0 \), (13) is obviously true.

Consider \( \|f(\alpha)\| > 0 \) below. Recall that for each complex number \( z \) and each \( \epsilon > 0 \)

\[
  e^z = \sum_{k = 0}^{\infty} \frac{z^k}{k!} \quad (28)
\]

\[
  \sum_{k = L}^{\infty} |z|^k < \epsilon \quad (29)
\]

where \( L \overset{\text{def}}{=} \max\{m, (1/\ln 2)(\ln(|2z|^m/m\epsilon))\} \), \( m = |2z| \), and \( [\cdot] \) denotes the largest integer (strictly) smaller than what is inside. Here, \( \ln(0)^{-1}/(-1)!\epsilon \) is taken to stand in for 0, so \( L = 0 \) in the case of \( z = 0 \).

With the uniform convergence of the series of (28) in \( \{z \in \mathbb{C} : |z| \leq 2\pi/\lambda\} \)

\[
  \int_0^{2\pi} e^{i(2\pi \sin \theta/\lambda)\alpha(\alpha) \cos(\phi - \alpha)} \tau_n(\alpha) \alpha(\alpha) d\alpha
\]

\[
  = \int_0^{2\pi} \left\{ \sum_{k = L}^{\infty} \frac{1}{k!} \left[ \frac{2\pi}{\lambda} \sin(\theta)\alpha(\alpha) \cos(\phi - \alpha) \right]^k \right\} \tau_n(\alpha) \alpha(\alpha) d\alpha
\]

\[
  = \sum_{k = L}^{\infty} \frac{1}{k!} \left( \frac{2\pi}{\lambda} \right)^k (\sin \theta)^k \alpha(\alpha) \tau_n(\alpha) \alpha(\alpha) \alpha(\alpha) d\alpha
\]

\[
  = \sum_{k = L}^{\infty} \frac{1}{k!} \left( \frac{e^{2\pi \sin \theta/\lambda} - \cos(\phi - \alpha)}{\lambda^2} \right)^k \tau_n(\alpha) \alpha(\alpha) d\alpha
\]

\[
  = \sum_{k = L}^{\infty} G_k(\theta) F_{k,n}(\phi)
\]

(30)
Hence, (13) may be obtained from (9) and (30) as follows

\[
B(\phi, \theta) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} f_n G_k(\theta)_{n,k}(\phi)
\]

\[
= \sum_{k=0}^{\infty} G_k(\theta) \sum_{n=-\infty}^{\infty} f_n F_k_{n,k}(\phi)_{n,k}
\]

\[
= \sum_{k=0}^{\infty} G_k(\theta) H_k(\phi)
\]

The second equality above is justified by this well known theorem in the theory of double series:

Let \( f(m, n) \) be a double sequence. Assume that

(i) \( \sum_{n=1}^{\infty} |f(m, n)| \) converges for each fixed \( m \)

(ii) \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m, n)| \) converges.

Then,

(a) \( \sum_{m=1}^{\infty} |f(m, n)| \) converges for each \( n \)

(b) \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) \) converges absolutely and equals \( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n) \).

To show that (i) is satisfied, note that

\[
\sum_{-\infty < k < \infty} |F_{k,n}(\phi)|^2
\]

\[
= \sum_{-\infty < k < \infty} \left| \int_0^{2\pi} \frac{r^2}{\lambda^2} \cos^2(\phi - \alpha) \tau_k(\alpha) s(\alpha) d\alpha \right|^2
\]

\[
= \int_0^{2\pi} \left| \int_0^{2\pi} \frac{r^2}{\lambda^2} \cos^2(\phi - \alpha) \tau_k(\alpha) s(\alpha) d\alpha \right|^2 d\phi
\]

\[
< \left( \frac{c}{\lambda^2} \right)^2
\]

(31)

where \( \ell \overset{\text{def}}{=} \int_0^{2\pi} s(\alpha) d\alpha \) denotes the total arc-length of the elliptical rim, and \( c \overset{\text{def}}{=} \max[a, b] \) symbolises the half-length of the major axis of the ellipse. The second equality above holds by the Parseval’s identity. The last strict inequality holds because the continuous integrand is strictly less than \( \left( c/\lambda \right)^2 \), except at no more than three points in the interval of integration. Hence, by the Cauchy–Schwarz inequality

\[
\sum_{-\infty < k < \infty} |f_k F_k_{n,k}(\phi)|
\]

\[
\leq \sqrt{\sum_{-\infty < k < \infty} |f_k|^2} \sqrt{\sum_{-\infty < k < \infty} |F_k_{n,k}(\phi)|^2}
\]

\[
< \|f(\alpha)\| \sqrt{\ell} \left( \frac{c}{\lambda} \right)^2
\]

(32)

The last inequality above is by (31).

To show that condition (ii) is satisfied, (32) gives

\[
\sum_{k=0}^{\infty} |G_k(\theta) F_k_{n,k}(\phi)| = \sum_{k=0}^{\infty} |G_k(\theta) F_k_{n,k}(\phi)|
\]

\[
= \|f(\alpha)\| \sqrt{\ell} \sum_{k=0}^{\infty} \frac{(2\pi)^2}{\lambda} \sin \theta |\frac{c}{\lambda}|^2
\]

\[
= \|f(\alpha)\| \sqrt{\ell} \sum_{k=0}^{\infty} \frac{(2\pi)^2}{\lambda} \sin \theta |\frac{c}{\lambda}|^2
\]

(33)

The last equality above is by (28). Thus, by (32), \( H_k(\phi) = \sum_{-\infty < k < \infty} f_k F_k_{n,k}(\phi) \) converges. By point (b) above, (13) is proved. [These same conclusions could have been proved alternatively as follows: Using a formula similar to (27), it may be proved that \( \sum_{k=0}^{\infty} |G_k(\theta) F_k_{n,k}(\phi)| < \infty \). Moreover, \( \sum_{-\infty < k < \infty} \sum_{k=0}^{\infty} |f_k||G_k(\theta) F_k_{n,k}(\phi)| < \infty \). Even though this alternative approach would be slightly shorter, the proof in the main body of this paper produces (32) and (33), which will be useful later in the analysis.]

\[
\Box
\]

12 Appendix 4: proof of Theorem 2

Let \( \varepsilon > 0 \) be arbitrarily given. Assuming \( \sin \theta \neq 0 \), (32) gives

\[
\sum_{k \geq K+1} |G_k(\theta) H_k(\phi)| < \|f(\alpha)\| \sqrt{\ell} \sum_{k \geq K+1} \frac{(2\pi)^2}{\lambda} \sin \theta |\frac{c}{\lambda}|^2
\]

\[
\leq \|f(\alpha)\| \sqrt{\ell} \sum_{k \geq K+1} \frac{(2\pi)^2}{\lambda} \sin \theta |\frac{c}{\lambda}|^2
\]

(34)

Recall the definition of \( K_\varepsilon \) in (14) and (15). By (29)

\[
\sum_{k \geq K+1} |G_k(\theta) H_k(\phi)| < \varepsilon
\]

If \( \sin \theta = 0 \), then \( \sum_{k \geq K+1} |G_k(\theta) H_k(\phi)| < \varepsilon \), because \( G_k(\theta) = 0 \) for all \( k \geq 1 \). Hence, (16) is proved.

By (13)

\[
|B(\phi, \theta) - \sum_{k=0}^{K} G_k(\theta) H_k(\phi)| \leq \sum_{k=K+1}^{\infty} |G_k(\theta) H_k(\phi)|
\]

Hence, (17) follows readily from (16). □
13 Appendix 5: proofs of Lemma 1 and Theorem 3

Proof of Lemma 1: The case of the infinite-term summation in (16) is easily settled as follows. First, observe that for any positive integer \( N \), (32) obviously remains valid if \( \sum_{n=-\infty}^{\infty} \) is replaced there throughout by \( \sum_{|n|<N} \). Then, the above proof for (16), with \( H_k(\phi) \) replaced by \( H_k^{(N)}(\phi) \) throughout, clearly shows that

\[
\sum_{k=K+1}^{\infty} |G_k(\theta)H_k^{(N)}(\phi)| < \epsilon
\]

(35)

That is, (16) remains valid with \( H_k(\phi) \) replaced by \( H_k^{(N)}(\phi) \), where \( N \) is any positive integer and \( K \) is any integer not less than \( K_\epsilon \).

The case of (17) is next analysed here. For each \( \eta > 0 \), let \( Q(\eta) \) be the smallest non-negative integer such that

\[
\|f'(\alpha)\|^2 - \sum_{|n|\leq Q(\eta)} |f_n|^2 \leq \frac{e^{-4\pi/A}}{\eta^2}
\]

(36)

The above exists, due to (8). Note that \( Q(\eta) \) decreases, as \( \eta \) increases. Consider (18) for every positive integer \( N > Q(\eta) \) and for \( k = 0, 1, \ldots \). Then, for any positive integer \( K \)

\[
\sum_{k=0}^{K} |G_k(\theta)H_k^{(N)}(\phi)| \leq \sum_{|n|>Q(\eta)} \left| G_k(\theta) \right| H_k(\phi) - H_k^{(N)}(\phi) \right| \leq \sum_{|n|>Q(\eta)} \left| G_k(\theta) \right| \sum_{|n|>Q(\eta)} \left| f_n \right| F_k(\phi) \right| \leq \sqrt{\frac{2}{\pi}} \sum_{|n|>Q(\eta)} |f_n|^2 \left| G_k(\theta) \right| \left| \frac{e^{-2\pi/A}}{\sqrt{\lambda}} \right| \left[ \text{by (31)} \right]
\]

\[
= \sqrt{\frac{2}{\pi}} \sum_{|n|>Q(\eta)} |f_n|^2 e^{2\pi/A} \left[ \leq \sqrt{\frac{2}{\pi}} \sum_{|n|>Q(\eta)} |f_n|^2 \right]^{1/2} \leq \eta
\]

(37)

The last inequality above is obtained by (13) and (37).

Define \( \delta \) as in (20), that is

\[
\delta \overset{\text{def}}{=} \eta - \|f'(\alpha)\| \leq \frac{e^{-4\pi/A}}{\eta^2}
\]

(38)

By (14), (15) and (29)

\[
\delta \in (0, \epsilon)
\]

(39)

By (34), \( \delta < \epsilon - \sum_{k=K+1}^{\infty} |G_k(\theta)H_k(\phi)| \). To have the smallest \( Q(\eta) \), pick the largest \( \eta \) (i.e. choose a specific \( \eta \) equal to \( \eta_0 = \delta \)). Denote \( Q(\eta) = Q(\eta_0) \). Then, \( Q \) is independent of \( k \). It follows from (38) that for all integer \( N > Q(\eta_0) \) independent of \( k \). This completes the proof.

Proof of Theorem 3: Direct substitution of (18) in (40) completes the proof.

14 Appendix 6: proof of Theorem 4

[If \( \eta = \epsilon \) in (38), and if the corresponding \( Q(\eta) \) is denoted as \( N^{(1)}(\epsilon) \)], it follows from (38) that for \( N > N^{(1)}(\epsilon) \) and \( K > K_\epsilon \)

\[
\frac{|B(\phi, \theta) - \sum_{n=-N}^{N} f_n \sum_{k=0}^{K} G_k(\theta)F_k(\phi)|}{|B(\phi, \theta) - \sum_{k=0}^{K} G_k(\theta)H_k^{(N)}(\phi)|} < 2\epsilon
\]

which is (23) for \( N^{(1)}(\epsilon) \). The reason for going through the present paragraph is that in general \( N^{(1)}(\epsilon) \) is larger than \( N_\epsilon \).

First, for every positive integer \( N \), the tail

\[
\Delta_N \overset{\text{def}}{=} \left| B(\phi, \theta) - \sum_{n=-N}^{N} f_n \int_{0}^{2\pi} e^{2\pi i \sin(\lambda/\phi) \cos(\phi-\alpha)} T_n(\alpha)(\phi) \, d\alpha \right|
\]
\[ \Delta_N \leq \sqrt{\epsilon} \sum_{|n|>N} |f_n|^2 \leq \epsilon \]  
(42)

Hence, for all positive integer \( K \geq K_e \) [i.e. the same \( K_e \) as defined in (14) and (15)] and all integer \( N \geq N_e \)

\[ |G_k(\theta)| \leq 2^{15-k} \sin (\theta) \]  
(43)
existence for all \( k \geq 13 \), because direct computation gives \( |j^{13}/(2\pi)^{13}| < 4 \), \( |j^{15}/(2\pi)^{15}| < 2 \), \( |j^{15}/(2\pi)^{15}| < 1 \), and \( |j^{15}/(2\pi)^{15}| < 1/2 \) if \( k \geq 13 \).

A ceiling of \( |F_{k,n}(\phi)| \) may be derived as follows: The Cauchy–Schwartz inequality implies that

\[ |F_{k,n}(\phi)| \leq \sqrt{\int_0^{2\pi} |F_{k,n}(\phi)|^2 d\phi} \sqrt{\int_0^{2\pi} |\tau_n(\phi)|^2 d\phi} \]

\[ \leq \sqrt{\frac{2\pi}{\lambda}} \int_0^{2\pi} |\tau_n(\phi)|^2 d\phi \sqrt{1} \]

\[ = \sqrt{\frac{\epsilon}{\lambda}} \lambda, \ \forall k, \forall n \]  
(44)

Furthermore, a ceiling of \( |H_k(\phi)| \) can be obtained as follows: For the non-trivial case of \( ||f(\alpha)|| > 0 \), it follows from (32) that

\[ |H_k(\phi)| < ||f(\alpha)|| \sqrt{\frac{\epsilon}{\lambda}} \]  
(45)

For the trivial case of \( ||f(\alpha)|| = 0 \), it is \( H_k(\phi) = 0 \) because \( f_n = 0 \) for all \( n \), and the strict inequality in (45) becomes an identity.

Consequently, from (41) and (34) as in (43), the following ceiling is obtained for the truncation error for the approach in Section 4

\[ \Delta_N \leq \sqrt{\epsilon} \sum_{|n|>N} |f_n|^2 \leq \epsilon \]  
(46)

16 Appendix 8: truncation analysis for any ‘smooth’ \( f(\alpha) \) and any ‘nice’ basis

The analysis in Section 4 (e.g. \( Q \) and \( N_e \)) depends on the convergence rate of the series \( \sum_{-\infty<n<\infty} |f_n|^2 \). Below is an alternative approach to produce results complementary to those in Section 4. In this alternative approach, \( N_e' \) (the counterpart of \( N_e \)) will make no reference to the convergence rate of the series \( \sum_{-\infty<n<\infty} |f_n|^2 \). However, this alternative analysis will require additional pre-conditions on the excitation function \( f(\alpha) \) regarding ‘smoothness’ and on the orthonormal basis, to yield uniform convergence in the subsequent (48) and (49) of the Fourier expansion in (47) below.

For a class \( S \) of excitation functions on the elliptical rim \( \mathcal{A} \), call an orthonormal basis \( \{\tau_n(\alpha), -\infty < n < \infty\} \) [for the space \( L^2([0, 2\pi], s(\alpha) d\alpha) \)] ‘S-nice’ if the Fourier series expansion of each \( f(\alpha) \in S \), with respect to the basis \( \{\tau_n(\alpha), -\infty < n < \infty\} \), converges uniformly to \( f(\alpha) \)
on $[0, 2\pi]$, that is
\[
f'(\alpha) = \sum_{|n| \leq N} f_n \tau_n(\alpha) \quad \text{(uniformly on } [0, 2\pi])
\]
(47)

From here onwards, consider a fixed class $S$ of excitation functions on the elliptical rim $\Lambda$, and a fixed ‘S-nice’ orthonormal basis $\{\tau_n(\alpha), -\infty < n < \infty\}$ for the space $L^2([0, 2\pi], \sin(\alpha) \, d\alpha)$. Therefore, for each given $f(\alpha) \in S$, there exists a sequence $E_{n,N}$, $N=0, 1, \ldots$, of positive numbers (which depend on the function $f(\alpha)$ and $N$) such that
\[
|f'(\alpha) - \sum_{|n| \leq N} f_n \tau_n(\alpha)| \leq E_{n,N} \quad \text{for all } \alpha \in [0, 2\pi]
\]
(48)

and
\[
\lim_{N \to \infty} E_{n,N} = 0
\]
(49)

Thus, for a fixed $f(\alpha) \in S$ and for an arbitrarily given $\varepsilon > 0$, let $N_0$ be chosen such that for all $N \geq N_0$
\[
E_{n,N} < \frac{\varepsilon}{\ell}
\]
(50)

where $\ell$ denotes the total arc-length of the elliptical rim $\Lambda$. Note that $N_0$ is independent of $\theta, \phi$, and the convergence rate of the series $\sum_{-\infty}^{\infty} |f_n|^2$.

**Theorem 6:** Let $S$ be a fixed class of excitation functions on the elliptical rim $\Lambda$, and $\{\tau_n(\alpha), -\infty < n < \infty\}$ be a fixed ‘S-nice’ orthonormal basis for the space $L^2([0, 2\pi], \sin(\alpha) \, d\alpha)$. Let $f(\alpha) \in S$. Let $K_{e,0}$ be given in (14), $K_{e,0}$ be given in (24), and $N_0$ be defined by (50). Then, for any integer $N \geq N_0$ and any integer $K \geq K_{e,0}$ and $K_{e,0}$, the following Parseval’s identity holds for every vector $x \in H$:
\[
\sum_{n=-N}^{N} |f_n|^2 = \sum_{n=-N}^{N} \left| \sum_{k=1}^{K} E_{n,k} \phi \right|^2
\]
(51)

and (12), respectively. Then, for all positive integer $K \geq K_{e}$ (or $K_{e,0}$) and all integer $N \geq N_0$
\[
|B(\phi, \theta) - \sum_{n=-N}^{N} f_n G_k(\theta) F_{k,n}(\phi)| < 2\varepsilon
\]
(52)

as in (43).}

Note that Theorem 6’s result differs from those in Section 4 in that $N_0$ depends on the rate of convergence of $\sum_{n=-\infty}^{\infty} |f_n|^2$, but $N_0$ makes no reference to that rate of convergence.

A quantitative ceiling of the resulting truncation error in $B(\phi, \theta)$ is given in the following theorem:

**Theorem 7:** For every pair of non-negative integers $K$ and $N$
\[
|B(\phi, \theta) - \sum_{n=-N}^{N} f_n G_k(\theta) F_{k,n}(\phi)| < 2\varepsilon\ell + \sqrt{\ell} \sum_{k=K+1}^{\infty} \left( \frac{2\pi |\sin \theta|}{\lambda} \right)^k
\]
(54)

where $E_{n,N}$ is given in (48).

The proof parallels that in Appendix 7.

### 17 Appendix 9: proof that $\Psi$ is an orthonormal basis of the elliptical rim

Recall that in a Hilbert space $H$ with an inner product $(\cdot, \cdot)$ and the induced norm $\| \cdot \|$, an orthonormal sequence $\xi_n$, $n \in \mathbb{N}$, forms an orthonormal basis for $H$ if and only if the following Parseval’s identity holds for every vector $x \in H$:
\[
\sum_{n=1}^{\infty} |(x, \xi_n)|^2
\]
Recall also that $e^{jna}/\sqrt{2\pi}$, $n \in \mathbb{Z}$, forms an orthonormal basis for the Hilbert space.
\[L^2([0, 2\pi], \alpha). \text{ Hence}
\]
\[
\langle \psi_n(\alpha), \psi_m(\alpha) \rangle \overset{\text{def}}{=} \int_0^{2\pi} \frac{e^{jna} - e^{-jma}}{\sqrt{2\pi} s(\alpha)} s(\alpha) \, d\alpha
\]
\[
= \int_0^{2\pi} \frac{e^{jna} - e^{-jma}}{\sqrt{2\pi} s(\alpha)} \, d\alpha
\]
\[
= \delta(n, m) \tag{55}
\]
where \(\delta(n, m)\) denotes the Kronecker delta function. Moreover, for every function \(g(\alpha) \in L^2([0, 2\pi], s(\alpha) \, d\alpha)\)
\[
\|g(\alpha)\|^2 \overset{\text{def}}{=} \int_0^{2\pi} |g(\alpha)|^2 s(\alpha) \, d\alpha
\]
\[
= \sum_{-\infty < n < \infty} \left| \int_0^{2\pi} g(\alpha) \sqrt{s(\alpha)} \frac{e^{-jna}}{\sqrt{2\pi}} \, d\alpha \right|^2
\]
(by the totality theorem)
\[
= \sum_{-\infty < n < \infty} \left| \int_0^{2\pi} g(\alpha) \sqrt{s(\alpha)} e^{-jna} \, d\alpha \right|^2
\]
\[
= \sum_{-\infty < n < \infty} |\langle g(\alpha), \psi_n(\alpha) \rangle|^2
\]
Hence, the sequence \(\Psi \overset{\text{def}}{=} \{\psi_n(\alpha) = e^{jna} / \sqrt{2\pi s(\alpha)}\}_{-\infty < n < \infty}\) is an orthonormal basis for the Hilbert space \(L^2([0, 2\pi], s(\alpha) \, d\alpha)\).