Algorithmic extensions of Su-Wong-Ho linear MMSE estimator for large-magnitude Levy-process phase-noise

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A new linear minimum-mean-square error (LMMSE) estimator has recently been proposed by Su, Wong and Ho to estimate phase-noise of (possibly) large magnitude, temporal non-stationarity, and Levy distribution (which includes the Wiener distribution as a special case). This estimator can handle many different degrees of latency. The estimator is adjustable to any number of taps, which may be pre-computed offline, based on only the signal-to-(additive-)noise ratio and the phase-noise’s characteristic function. This pre-computation requires no matrix inversion. The above estimator is algorithmically extended for more flexibility in latency, to select the optimum estimator-tap support-window from a wider data-observation window, and to handle newly arrived samples in a computationally efficient manner.

Introduction: The phase-noise \( \theta_k \), at the \( k \)th time-point, is defined as the difference between the phase of the received carrier sinusoid and the phase of the receiver’s local oscillator. This phase-noise multiplicatively corrupts an information-signal through the stochastic process, \( e^{j\theta_k} \). High-magnitude non-stationary phase-noise may be estimated by the recently advanced Su-Wong-Ho phase-noise estimator [1].

After reviewing the Su-Wong-Ho LMMSE estimator, this Letter offers extensions along these three directions: it extends the Su-Wong-Ho phase-estimator’s latency from \( L \in \{ 0, 1, \ldots, |Q - 1/2| \} \) or [3], to \( L \in \{ 0, 1, \ldots, |Q - 1| \} \) where \( Q \) denotes the number of taps in the estimator. In the above, \(|a|\) refers to the largest integer not exceeding the real number \( a \). It investigates this issue: if more than \( Q \) temporally contiguous observables are available to a Q-tap estimator, which \( Q \) observables should be chosen to minimise the estimator’s mean square error? Then it addresses the issue: for a Q-tap estimator \( e^{j\theta_k} \) at a set latency of \( L \), what happens if the observation-window and the set of to-be-estimated time-samples both slide rightward along the discrete-time axis? How may this right-shifted Su-Wong-Ho LMMSE estimator’s weights be calculated from the original ‘presift’ weights?

Review of Su-Wong-Ho phase-noise estimator [3]: Consider a baseband observable datum \( n_k = A \exp(j \theta_k) + n_k \) at the \( k \)th uniformly spaced time-sampling instant, where \( A \) denotes the signal’s constant (and possibly unknown) magnitude, \( n_k \) represents the 4th time-sample’s unknown zero-mean complex-value white noise of a (possibly unknown) variance \( E[n_k^2] = \sigma^2_k \). The signal-to-noise (SNR) power ratio, \( (A/\sigma_k)^2 \), is either an a priori known or estimated in an earlier step. Moreover, \( \theta_k \) symbolises the unknown and to-be-estimated phase-noise at the \( k \)-th time-instant. The discrete-time sequence \( \{ \theta_k, \forall k \in \{ 1, 2, \ldots \} \} \) equals a symmetric Levy process sampled at time \( k = 1, 2, \ldots \), with a known priori \( E[e^{j\theta_k}] \). The phase-noise random sequence \( \{ \theta_k, \forall k \geq 1 \} \) is independent of the additive noise random sequence \( \{ n_k, \forall k \geq 1 \} \). The above has not required \( |\theta_k| \ll 1 \).

The Su-Wong-Ho estimator [3] is

\[
e^{j\hat{\theta}_k} = \frac{1}{A} \sum_{q=1}^{Q} w_q^{(k)} e^{jQ-L-q} \tag{1}
\]

for a particular latency of \( L \) and a particular estimator-length of \( Q \geq 2L + 1 \), where

\[
w_q^{(k)} = \frac{A^2}{\sigma^2_k} - \alpha L - 1 - \frac{1 - c^2}{c} \tag{2}
\]

for \( 1 \leq q < Q - L \) and

\[
w_q^{(k)} = \frac{\alpha_q}{c^{q+1}} \quad \text{for} \quad Q - L < q \leq Q \tag{3}
\]

where \( \beta = 1/c^2, \delta_k = 1/c, \alpha_k = 1/\beta - \alpha_{k-1} \geq 1 \forall \). This is the linear estimator with the least ‘mean-square error’, \( \text{MSE}(L, w) = [e^{j\hat{\theta}_k} - e^{j\theta_k}]^2 \).

Generalisation of Su-Wong-Ho phase-estimator to any \( Q > L \geq 0 \): By definition,

\[
\text{MSE}(L, w) = 1 + \frac{\sigma^2_k}{A^2} \left[ \sum_{q=1}^{Q} w_q^{(k)} \right]^2 - 2 \sum_{q=1}^{Q} w_q^{(k)} e^{jQ-L-q} + \sum_{q=1}^{Q} \sum_{m=1}^{Q} w_q^{(k)} w_m^{(k)} e^{j(Q-L-q)} + \sum_{q=1}^{Q} \sum_{m=1}^{Q} w_q^{(k)} w_m^{(k)} e^{jQ-L-q} \tag{4}
\]

for \( Q > L \geq 0 \).

Define \( \hat{w} = [\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_Q] = [w_1, w_2, \ldots, w_Q] \). Hence,

\[
\sum_{q=1}^{Q} w_q^{(k)} e^{jQ-L-q} = \sum_{q=1}^{Q} w_{Q+1-q} e^{-jQ-L+q} = \sum_{q=1}^{Q} w_{Q+1-q} e^{j(Q-L-q)} = \sum_{q=1}^{Q} w_{Q+1-q} e^{jQ-L-q} \tag{5}
\]

Therefore,

\[
\text{MSE}(Q - L - 1, \hat{w}) = 1 + \frac{\sigma^2_k}{A^2} \sum_{q=1}^{Q} w_q^{2} - 2 \sum_{q=1}^{Q} w_q^{(k)} e^{jQ-L-q} + \sum_{q=1}^{Q} \sum_{m=1}^{Q} w_q^{(k)} w_m^{(k)} e^{jQ-L-q} \tag{6}
\]

\[
\text{MSE}(L, w) = \sum_{q=1}^{Q} w_q^{(k)} e^{jQ-L-q} \tag{7}
\]

As (5) holds \( \forall Q > L \geq 0 \) and \( \forall w \in \mathbb{C}^Q \), it is true that \( w_1^{Q}, \ldots, w_Q^{Q} \) minimises \( \text{MSE}(L, w) : w \in \mathbb{C}^Q \), i.e., \( w_1^{Q}, \ldots, w_Q^{Q} \) minimise \( \text{MSE}(Q - L - 1, \hat{w}) : w \in \mathbb{C}^Q \). The Su-Wong-Ho phase-estimator has thus been extended to \( \forall Q > L \geq 0 \).

How to place estimator-taps-window within larger data-observation window: The Su-Wong-Ho LMMSE estimator’s \( Q \)-tap weights are independent of \( k \), but depend on the latency \( L \). With the most recent observable at the \( m \)th sampling instant, obviously it must be true that \( L \leq M - k \). If more than temporally contiguous observables are available to the \( Q \)-tap Su-Wong-Ho estimator, guidelines are given below to pick which \( Q \) contiguous observables for the estimator. This Section will rigorously prove that for a preset \( Q \), the \( Q \)-tap data-window should centre around the \( q \)th time-sample as much as possible.

Theorem 4.1: Suppose that \( \{ \theta_k, k \leq M \} \) is to be estimated using a continuous \( Q \)-tap observation-window taken from \( \{ n_k, m = 1, 2, \ldots, M \} \). The following will minimise the MSE:

(i) if \( k \in \{ Q - 1/2, \ldots, M - Q - 1/2 \} \), set \( L = |Q - 1/2| \).

(ii) if \( k > M - Q - 1/2 \), set \( L = M - k \).

(iii) if \( k < Q - 1/2 \), set \( L = Q - k \) and \( e^{j\hat{\theta}_k} = 1/A \sum_{q=1}^{Q} w_q^{(k)} \).

Proof: Define

\[
\text{MMSE}(L) = \inf_w \text{MSE}(L, w) = \text{MSE}(L, w^{(Q-L)}) \tag{8}
\]

where \( w^{(Q-L)} = [w_1^{(Q-L)}, w_2^{(Q-L)}, \ldots, w_{Q-L}^{(Q-L)}] \) denotes the LMMSE weights at latency \( L \). Let \( L^* \) denote the ‘best’ latency over \( L = 0, 1, \ldots, Q - 1 \), in the sense of

\[
\text{MMSE}(L^*) = \min_L \text{MMSE}(L) \tag{9}
\]

As shown in [3], the LMMSE weights \( w_1^{Q-L}, w_2^{Q-L}, \ldots, w_{Q-L}^{Q-L} \) must be positive real numbers, so the following will consider only \( w_q \in (0, \infty), \forall q = 1, \ldots, Q \). These LMMSE-estimator weights satisfy \( \partial^2/\partial w_q \text{MMSE}(L, w) = 0, \forall q \):

\[
w_q^{(Q-L)} = \frac{\sigma^2_k}{A^2} \sum_{m=1}^{Q-L} w_m^{(Q-L)} e^{-jQ-L-q} = e^{jQ-L-q} \tag{10}
\]

Equivalently,

\[
\sum_{m=1}^{Q-L} w_m^{(Q-L)} e^{-jQ-L-q} = e^{jQ-L-q} \frac{\sigma^2_k}{A^2} \tag{11}
\]

As (5) implies

\[
\text{MSE}(Q - L - 1) = \text{MMSE}(L), \forall L = 0, 1, \ldots, Q - 1 \tag{12}
\]

it would suffice to consider only \( L \leq |Q - 1/2| \).
Suppose $L < |Q - 1/2|$, or equivalently $Q \geq 2L + 3$. Then,

$$\text{MSE}(L) - \text{MSE}(L + 1) = \frac{Q}{\sum_{q=1}^{Q} (w_{q}^{(L+1,0)})^2 (\alpha_{q}^2)} = 2 \left( 1 - w_{Q,L-1} (\alpha_{Q,L-1})^2 \right) \frac{a_{Q,L-1}^2}{A^2}$$

$$+ \sum_{q=1}^{Q} w_{q}^{(L,0)} \left( w_{q}^{(L+1,0)} - w_{q}^{(L,0)} \right) \frac{\alpha_{q}^2}{A^2}$$

$$- \left\{ \sum_{q=1}^{Q} (w_{q}^{(L+1,0)})^2 (\alpha_{q}^2) \right\} - 2 \left( 1 - w_{Q,L-1} (\alpha_{Q,L-1})^2 \right) \frac{a_{Q,L-1}^2}{A^2}$$

$$+ \sum_{q=1}^{Q} w_{q}^{(L+1,0)} \left( w_{q}^{(L+1,0)} - w_{q}^{(L,0)} \right) \frac{\alpha_{q}^2}{A^2}$$

$$= \frac{\left( w_{Q,L} - w_{Q,L-1} \right) (\alpha_{Q,L-1})^2}{A^2}$$

$$= \frac{\left( \beta - \alpha_{Q,L-1} \right) \left( \beta - \alpha_{Q,L-2} - \alpha_{Q,L-1} \right)}{c} \frac{1 - c^2}{c} > 0$$

The above (12) is obtained by setting $q = Q - L$ in (9) and setting $q = Q - L - 1$ in (10). That gives

$$\sum_{q=1}^{Q} w_{q}^{(L,0)} (\alpha_{q}^2) \left( w_{q}^{(L+1,0)} - w_{q}^{(L,0)} \right) \frac{\alpha_{q}^2}{A^2}$$

To prove (13), it would be sufficient to show that $(\alpha_{Q,L-2} - \alpha_{L}) > (\alpha_{Q,L-2} - \alpha_{Q,L-1})$. Towards this end, note that $\alpha_{Q,L-1} = 1/\beta - \alpha_{Q,L-1}$ and $\alpha_{Q,L-2} = 1/\beta - \alpha_{Q,L-2}$. So, $(\alpha_{Q,L-2} - \alpha_{L}) = (1/\beta - \alpha_{Q,L-2} - (1/\beta - \alpha_{Q,L-1}))$. Because $Q \geq 2L + 3$, and $\alpha_{Q,L-2}$ both strictly decrease with $m$, it holds that $Q - L < 1$ and $L > 0$. This proves (13) and completes the proof of the theorem. □

Recursive update of Su-Wong-Ho estimator weights to incorporate new sample: For a $Q$-tab estimator, what happens if both the $Q$-tab observation-window and the to-be-estimated time-sample ($e^{j\theta}$) slide to the right along the discrete-time axis, but $L$ remains the same? How may the correspondingly right-shifted Su-Wong-Ho LMMSE estimator’s weights be related to the original ‘pre-shift’ Su-Wong-Ho LMMSE estimator’s weights? The LMMSE estimator’s $Q$-taps $\{w_{q}^{(L,0)}\}$, that respectively weighting the observables $\{r_{q+L-Q+q}^{}, q = 1, \ldots, Q\}$ in $e^{j\theta}$, those taps would be applicable also in $e^{j\theta+\pi}$ to weight $\{r_{q+L-Q+q+1}, q = 1, \ldots, Q\}$ after the aforementioned ‘shifting’. This is because

(i) between $\theta_{q+L}$ and $\theta_{q+L+1}$ is a time lapse equal to that between $\theta_{q}$ and $\theta_{q+1}$, and

(ii) the phase noise is modelled to have stationary independent increments.

The weights are thus independent of $k$, even though $\{\theta_{q}\}$ and $\{e^{j\theta}\}$ are non-stationary. This independence of $k$ is because the estimator multiplies $r_{k}$ with $r_{m}$. $\forall m$.

Conclusion: The Su-Wong-Ho phase-estimator of [3] was for a limited range of latency, and for a pre-specified number of taps equal to the observation-window size. That estimator is herein extended to accommodate a wider range of latency, to select the optimum estimator-taps window from a wider observation-window, and to handle newly arrived samples.

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