Spatial correlation-coefficient across a receiving sensor-array – accounting for propagation loss

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In the open literature on the geometric modelling of wireless radio wave land mobile propagation channels, this reported work is the first to attempt to account for the propagation power loss of the multipaths, in deriving closed-form expressions for an uplink received-signal’s spatial-correlation-coefficient function across the aperture of a base-station antenna-array.

Introduction: The development of ‘smart antenna’ systems would be aided by a simple rule-of-thumb, to estimate the inter-antenna spacing needed for a required level of the spatial correlation coefficient. This Letter offers such a simple rule, in terms of the transmitter’s arrival angle to the receiving antennas.

Proposed geometric model: Consider a (mobile) transmitter located at \( z_{\text{TM}} = [z_{x\text{TM}}, z_{y\text{TM}}] \) on the Cartesian \( x\text{-}y \) plane emitting omnidirectionally (see Fig. 1). The transmitted signal bounces off any scatterer (located at \( z = [z_x, z_y] \)) in the surrounding environment, producing a multitude of propagation-multipaths towards diverse directions. This model assumes \( S \) number of such multipaths to be arriving at each receiving antenna, with each multipath representing one ‘bounce’ off a different scatterer in the channel.

![Spatial correlation-coefficient across a receiving sensor-array – accounting for propagation loss](image)

Fig. 1 Geometry relating transmitter, scatterers, and base-station antennas

Each multipath suffers propagation-path loss, which worsens as the area of the wavefront increases with propagation distance. The propagation-loss model is described as:

\[
g(\mathbf{z}) = |\mathbf{z} - z_{\text{TM}}|^\alpha, \quad \alpha \in \{1, 2, 3, \cdots \} \tag{1}
\]

where \( \alpha \) denotes the path-loss exponent and implicitly regulates the spatial extent within which scatterers are significant. This propagation path loss model in (1) contrasts with customary modelling of \( g(\mathbf{z}) \) as independent of \( \mathbf{z} \) (or equivalently, \( n = 0 \)).

The scatterers’ spatial locations \( \mathbf{z} = [z_x, z_y] \) are modelled as drawn from an homogeneous Poisson spatial point process \( \Pi(B) \) indexed on subsets of a two-dimensional Cartesian plane \( R^2 \). For any spatial region \( B \), the random number \( \Pi(B) \) of points in \( B \) is distributed according to a Poisson law with parameter \( E[\Pi(B)] = A(B) \); and \( \Lambda(B) \) gives the expected number of scatterers in \( B \). For an homogeneous Poisson process, a positive scalar \( A \) suffices to denote the point process intensity. This spatial homogeneity would not restrict the spatial extent of the scatterers’ locations. Moreover, only the statistical expectation of the spatial density of the scatterer-field has been specified, not the scatterers’ actual locations.

Each scatterer is modelled as an omnidirectional lossless transmitter of any incoming signal from the transmitter, thereby producing a multipath towards each omnidirectional receiving-antenna. All multipaths, reflecting off a particular scatterer located at \( \mathbf{z} \), have an initial phase \( \varphi(\mathbf{z}) \) uniformly distributed over \((-\pi, \pi] \). \( \varphi(\mathbf{z}) \) is statistically independent of the spatial stochastic point process \( \Pi(\mathbf{z}) \) and \( \varphi(\mathbf{z}) \) are statistically independent for two scatterers located at \( \mathbf{z} \neq \mathbf{z} \).

Deriving the spatial correlation coefficient: The transmitted signal bounces off a scatterer at \( \mathbf{z} \), producing a multipath towards receiving-antenna 1 located at \( \mathbf{z}_{\text{BS1}} = (0, 0) \), and another multipath towards antenna 2 at \( \mathbf{z}_{\text{BS2}} = (d_{\text{p}}, 0) \). These two multipaths’ complex-value amplitudes are multiplied by the channel-propagation (complex-valued) coefficients of \( g(z) = g(z)e^{j\varphi(z)} \), and \( c_2(z) = g_2(z)e^{j\varphi_{2}(z)} \). The ratio \( \Delta_{p}(z) \) represents the phase difference between the two transmitted multipaths at \( \mathbf{z}_{\text{BS}} = (0, 0) \) against at \( \mathbf{z}_{\text{BS}} = (d_{\text{p}}, 0) \). This \( \Delta_{p}(z) \) depends on the distances traversed by the multipaths reaching the two sensors, and differs from the previously defined \( \varphi(\mathbf{z}) \). The following derives an expression for this \( \Delta_{p}(z) = (2\pi/\lambda) [s_1(z) - s_2(z)] \), where \( \lambda \) denotes the wireless carrier wavelength, \( z = [r \cos(d\varphi(z)), r \sin(d\varphi(z))] \) is in the polar co-ordinates, with the polar axis coinciding with the Cartesian \( x \)-axis. Moreover, \( s_i(z)|z^2 = [a_i(z)]^2 + d_i^2 + 2a_i(z)d_i \cos(\alpha_i(z)) \) and \( d_i = ||\mathbf{z}_{\text{BS}} - \mathbf{z}_{\text{BS}}|| \), for \( i = 1, 2 \). Furthermore, \( \alpha_i(z) = |z - z_{\text{BS}}| = \sqrt{(z_x - z_{\text{MS},i})^2 + (z_y - z_{\text{MS},i})^2} \). Because \( \Lambda \ll 1/d_{\text{p}} \),

\[
\Delta_{p}(z) = \frac{2\pi}{\lambda} \left[ \sqrt{z_x^2 + z_y^2} - \sqrt{z_x^2 + d_{\text{p}}^2 + z_y^2} \right]
\]

\[
\approx 2\frac{d_{\text{p}}}{\lambda} \frac{z_y}{\sqrt{z_x^2 + z_y^2}} = 2\frac{d_{\text{p}}}{\lambda} \cos(d\varphi(z)) \tag{2}
\]

The first approximation above is by the Taylor series expansion, \( \sqrt{1 + x} \approx 1 + (x/2) \). The second approximation omits all higher-than-first-order terms from an infinitesimal expansion with respect to \( (d_{\text{p}}/\sqrt{z_x^2 + z_y^2}) \), which has been assumed to be \( \ll 1 \). In other words, the only geometric requirement now is \( d_{\text{p}} \ll r = \sqrt{z_x^2 + z_y^2} \).

Vector-sum all multipaths arriving at antenna 1 from all scatterers on \( A \subseteq R^2 \), to obtain \( C_1 = \int_A g(z)g(\mathbf{z})e^{j\varphi(z)} \Pi(\mathbf{z})d\mathbf{z} \). Similarly for antenna 2,

\[
C_2 = \int_A g(z)g(\mathbf{z})e^{j\varphi(z)} \Pi(\mathbf{z})d\mathbf{z} \tag{3}
\]

Because \( \varphi(\mathbf{z}) \) and \( \Pi(\mathbf{z}) \) are statistically independent and because \( g(\mathbf{z}) \) is uniformly distributed, stochastic integration gives \( E[C_1] = \int_A E[e^{j\varphi(z)}]E[\Pi(\mathbf{z})d\mathbf{z}] = 0 \), and \( E[C_2] = \int_A E[e^{j\varphi(z)}]E[\Pi(\mathbf{z})d\mathbf{z}] = 0 \). The preceding line holds because \( \Delta_{p}(z) \) depends statistically only on \( \Pi \). Hence, the spatial covariance and the spatial correlation both equal:

\[
E(d_{\text{p}}^2) = E[C_1]E[C_2^*] = \Lambda \int_A E[e^{j\varphi(z)}E[\mathbf{z}]]d\mathbf{z}
\]

To facilitate subsequent derivation, approximate (1) as \( g(\mathbf{z}) \\approx [|z - z_{\text{BS}}|^2 + \delta^2]^{\alpha/2} \), for \( \delta = 0 \). This approximation converges to (1) as \( \delta \rightarrow 0 \). This constant \( \delta > 0 \) is included to preclude any non-integrable pole at zero in the subsequent steps. The limit \( \delta \rightarrow 0 \) will be taken later, such that \( \delta \) will disappear from the final formula. Hence, \( \delta \) is not a model parameter. Along with (2), rewrite (3) as

\[
E(d_{\text{p}}) = \Lambda \int_{\mathcal{B}} \frac{1}{(z - z_{\text{BS}})^{\alpha} + \delta^2} \cos(d\varphi(z))d\mathbf{z} \tag{4}
\]

where \( \delta \) denotes the angle of the position-vector \( z = [r \cos(d\varphi), r \sin(d\varphi)] \) in the polar co-ordinates.

Express \( \mathbf{z}_{\text{BS}} = [r_{\text{BS}} \cos(d\varphi), r_{\text{BS}} \sin(d\varphi)] \) in the signed-polar co-ordinates, which are inter-related to the Cartesian co-ordinates through \( z_{\text{BS}} = r_{\text{BS}} \cos(d\varphi), r_{\text{BS}} = \sqrt{z_{x_{\text{BS}}}^2 + z_{y_{\text{BS}}}^2}, \varphi_{\text{BS}} = \text{arccos}(z_{x_{\text{BS}}}/\sqrt{z_{x_{\text{BS}}}^2 + z_{y_{\text{BS}}}^2}), \forall \mathbf{z}_{\text{BS}} \in (-\infty, \infty), \forall \varphi_{\text{BS}} \in [0, \pi] \), where \( \text{sgn}(x) = \{1, x \geq 0\} \) and \( \text{sgn}(x) = \{-1, x < 0\} \) and \( \int_0^\infty \text{sgn}(x) e^{ax} dx = \left\{ \begin{array}{cc} 0, & x > 0 \\infty, & x \leq 0 \end{array} \right. \). Hence, rewrite
where the preceding equality comes from [4] (equation (5) in section 3.252).

Set \( d_{\text{dp}} = 0 \) in \( \Delta_{x}(z) \) to obtain the variance (i.e., the power),

\[ E[|C_{1}|^{2}] = E[|C_{2}|^{2}] = \sigma^{2} = \Lambda \int A \phi(x) \phi(x)^{*} \, dx. \]

Using the polar coordinates,

\[ \sigma^{2} = 2 \pi \Lambda \int_{0}^{\infty} \frac{r \, dr}{r^{2} + \delta^{2}} = \frac{2 \pi \Lambda}{n - 1} \delta^{2n-2} \]

Divide (5) by (6), to give the spatial correlation coefficient,

\[ \rho(d_{\text{dp}}) = \frac{C(d_{\text{dp}})}{\sigma^{2}} = \left( \frac{n - 1}{2} \right) \left( \frac{2(n - 3)!}{(n - 2)!} \right) \left( \frac{r_{\text{MS}}}{\delta} \right)^{2-2n} \Lambda^{2} \frac{\delta^{2}}{r_{\text{MS}}} \phi_{x_{\text{MS}}} \]

where \((2m)! = 2 \cdot 4 \cdot 6 \cdots \cdot (2m), \) and \((2m - 1)! = 1 \cdot 3 \cdot 5 \cdots \cdot (2m - 1), \) for any natural number \(m;\)

\[ A(x, y, \phi_{\text{MS}}) = \int_{0}^{\pi} e^{i2\pi r_{\text{MS}} \cos \phi} \frac{\cos(\phi - \phi_{\text{MS}})}{[\sin^{2}(\phi - \phi_{\text{MS}})]^{2n-1/2}} \, d\phi \]

with \(x = (d_{\text{dp}}/\lambda), \quad y = (\delta/r_{\text{MS}})^{2};\)

Set \((\delta/r_{\text{MS}}) \ll 1\) as \(\delta \gg 0,\) limit \(\phi_{\text{MS}} \in [0, \pi];\) and define a small \(\epsilon(y)\) such that \(\epsilon(y)/\sqrt{y} \ll 1,\) nonetheless. These give

\[ A(x, y, \phi_{\text{MS}}) = \left[ e^{2\pi r_{\text{MS}} \cos \phi} + \int \phi(x) \phi_{\text{MS}}(y) \right] e^{2\pi m \cos \phi} \frac{\cos(\phi - \phi_{\text{MS}})}{[\sin^{2}(\phi - \phi_{\text{MS}})]^{2n-1/2}} \, d\phi \]

\[ = e^{2\pi m \cos \phi} \left[ e^{2\pi m \cos \phi} \int_{-y}^{y} \frac{d\phi}{[\cos(\phi - \phi_{\text{MS}})]^{2n-1/2}} + R(y) \right] \]

\[ = e^{2\pi m \cos \phi} \left[ e^{2\pi m \cos \phi} \int_{-y}^{y} \frac{d\phi}{(\cos^{2}(\phi - \phi_{\text{MS}}))]^{2n-1/2}} + R(y) \right] \]

From [1] (equation (10) in section 3.252 and equation (1) in section 8.756),

\[ c_{v} = \int_{-\infty}^{\infty} \frac{dt}{(t^{2} + 1)^{2n-1/2}} = 2^{n-1} \Gamma(n) \]

\[ \times B(1, 2n - 2)P_{-1/2, n}(0) = 2^{n-1} \Gamma(n) \frac{1}{(2n - 3)!} \]

where \(\Gamma(\cdot)\) refers to the gamma function, \(B(\cdot, \cdot)\) symbolises the beta function, and \(P_{\alpha, \beta}^{\gamma}(\cdot)\) denotes the associated Legendre function of the first kind. Furthermore,

\[ |R(y)| \leq 2 \int_{\epsilon(y)}^{\pi} \frac{d\phi}{[\sin^{2}(\phi - \phi_{\text{MS}})]^{2n-1/2}} \leq \frac{2\pi}{[\sin^{2}(\phi - \phi_{\text{MS}})]^{2n-1/2}} \leq \frac{2\pi}{\epsilon^{2n-1}(y)} \]

which needs be much smaller than the first summand. Hence, \(\epsilon(y)/\sqrt{y} \ll 1.\) Hence, choosing \(\epsilon(y)\) such that \(\epsilon(y) \gg \frac{y^{n-1/2}}{2n-1} - \frac{1}{2n-1} \frac{2\pi}{\epsilon^{2n-1}(y)} \)

With \(2n - 2)! = 2^{n-1}(n - 1)!,\) and substituting (8) in (7),

\[ \rho(d_{\text{dp}}) = \left( \frac{r_{\text{MS}}}{\delta} \right)^{2-2n} \left( \frac{2n - 1}{2(2n - 3)!} \right) \left( \frac{2n - 2}{2(2n - 3)!} \right) \left( \frac{\epsilon^{2n-1}(y)}{\epsilon^{2n-1}(y)} \right) \]

The above is independent of \(\Lambda.\) From (9), the magnitude of the spatial correlation coefficient function equals

\[ \text{Re}[\rho(d_{\text{dp}})] = \left( \frac{2\pi d_{\text{dp}}}{\lambda} \cos(\phi(z_{\text{MS}})) \right) \]

Conclusion: This derived formula is independent of the power-loss exponent \(n,\) which implicitly regulates the spatial extent within which a scatterer must be located, for the scatterer to be a notable retransmitter. This \(n\) assigns increasing influence to the retransmitted propagation-path of a scatterer, as that scatterer lies closer to the transmitter. As \(n\) increases, this partiality for close-by scatterers becomes more pronounced.

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