Copula – to model multi-channel fading by correlated but arbitrary Weibull marginals, giving a closed-form outage probability of selection-combining reception

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Abstract: This study introduces the mathematical paradigm of ‘copula’ to the technical field of channel modelling. Copula-based modelling allows each sensor of its distinct set of fading parameters, and allows cross-correlation among various sensors’ parameter sets. Hence, the branch statistics may be non-identical over various branches, yet cross-correlated between branches. This scenario is realistic especially for sensors of different polarisations/constructions and/or for sensors at widely separate locations – as increasingly common for wireless transceivers. Despite such versatility in modelling, this copula-based multivariate fading distribution may yet be so simple in mathematical form that the system’s outage probability can be analytically derived in a closed form, free of any infinite-term summation and free of any unsolved integration.

1 Introduction

In fourth- and fifth-generation wireless communications, transceivers deploy multiple antennas. Across these antennas, the channel’s fading is often cross-correlated; yet for each antenna, the fading statistics may be unique. That is, multi-channel fading, modelled stochastically as a multivariate probability distribution, would have marginal univariate distributions that are arbitrarily different among themselves yet cross-correlated.

For a receiver using branch-selection reception, how may the outage probability depend on the various antennas’ various fading parameters? For cross-correlated multivariate Weibull fading, the existing literature has answers to this important question, but only in open forms, involving nested infinite summations and/or unsolved integrals. These complicated expressions, admittedly derived with much mathematical acumen, are difficult for even a computer to numerically evaluate, let alone being qualitatively revealing to a system engineer’s naked eyes.

This paper will derive the above-mentioned outage probability in closed-form, as an algebraic function of real exponents, requiring no integral and no infinite-term summation, while allowing a more flexible class of cross-correlated multivariate Weibull fading than in the current paper. This new derivation is based on the probability concept of ‘copula’, which allows the statistical dependence to be specified across random scalars (describing the individual antennas’ fading), even as these random scalars take on arbitrary univariate distributions. This ‘copula’ concept is well established in probability, but apparently unknown to the channel-fading literature in wireless communication engineering.

1.1 Faded signal envelope as Weibull distributed

The Weibull distribution has been found empirically in [1–16] to be descriptive of the received signal’s magnitude, for some wireless propagation scenarios indoor/outdoor/vehicle-to-vehicle/on-body and at various frequency bands.

A Weibull random variable \( Z_f \), by definition, has a probability density function of

\[
\frac{\beta_f}{\Omega_f} \left( -\frac{z_f}{\Omega_f} \right)^{\beta_f-1} \exp \left( -\frac{z_f}{\Omega_f} \right), \quad \text{if} \ z_f \geq 0, \\
0, \quad \text{if} \ z_f < 0,
\]

with a ‘scale parameter’ of \( \Omega_f = E[Z_f^\beta_f] > 0 \) and a ‘shape parameter’ of \( \beta_f > 0 \). The Weibull distribution’s ‘shape parameter’ \( \Omega_f \) is called the ‘Weibull fading parameter’ on p. 3609 of [17].

The corresponding cumulative distribution function (CDF) equals \( F_f(z_f) = 1 - \exp(-z_f^{\beta_f}/\Omega_f), \quad z_f \geq 0 \). This Weibull random variable \( Z_f \) generalises a Rayleigh random variable \( R_f \), via the relationship \( Z_f = R_f^{\beta_f} \).

A Weibull random variable can model any one sensor’s fading phenomenon. Indeed, if the \( k \)th sensor’s fading is statistically dissimilar to that at the \( r \)th sensor, the \( k \)th sensor’s random variable \( Z_k \) would have an \( \Omega_k \neq \Omega_r \) and/or a \( \beta_k \neq \beta_r \). For such a wireless receiver equipped with more than one sensor, the system’s efficacy would depend critically on how the various sensors’ fading is cross-correlated, statistically speaking. Hence, such cross-sensors cross-statistics needs to be part of the channel model, in order to link the above-mentioned univariate probability distributions for the various individual sensors. That is, a correlated multivariate distribution needs to be mathematically constructed, to encompass all sensors’ fading, in order to account for the cross-correlation among the marginal univariate distributions in (1). Toward this end, the mathematical paradigm of ‘copula’ (apparently unknown to the channel-modelling literature) will be shown in this paper to be a very versatile tool.

1.2 Selection-combining (SC) diversity-reception and its outage probability

If the \( r \)th sensor’s signal envelope \( Z_r \) is Weibull distributed at \( \Omega_r \) and \( \beta_r \) that sensor’s ‘per-symbol signal-to-noise ratio’ (SNR) \( \gamma_r = Z_r^2 E_r / N_0 \) would also be Weibull distributed, but at a ‘scale
parameter’ of $\beta_i = \beta/2$ and a ‘shape parameter’ of $\gamma_i = \Omega_i E_i/(N_0)$\(^{\beta_i}\). (Please see [17], section III, paragraph two.) Here, $E_i$ denotes the signal power and $N_0$ symbolises the noise power. For a multi-sensor receiver, ‘selection combining’ (SC) reception would process the data from only the one sensor with the highest SNR, but would discard all remaining sensors’ data. Hence, the selection-combiner’s output would have an effective SNR of $\gamma_{\text{SC}} = \max(\gamma_1, \gamma_2, \ldots, \gamma_L)$. Please see (30), (31), and (61) in [17].

The outage probability ($P_{\text{out}}(r_{th})$) is defined as the probability of $\gamma_{\text{SC}}$ falling below the specified threshold of $r_{th}$.

\[
P_{\text{out}}(r_{th}) = P_{\text{yc}}(r_{th}) \\
= \text{Prob}(\gamma_{\text{SC}} \leq r_{th}) \\
= \text{Prob}(\gamma_1 \leq r_{th}, \ldots, \gamma_L \leq r_{th}) \\
= F_{\gamma_1, \gamma_2, \ldots, \gamma_L}(r_1, r_2, \ldots, r_L) \\
= \int_0^r \cdots \int_0^r f_{\gamma_1, \gamma_2, \ldots, \gamma_L}(r_1, r_2, \ldots, r_L) \, dr_L \cdots dr_1
\]  

(2)

Mathematically, (2) shows the formula of $P_{\text{out}}(r_{th})$, where $F_X(x)$ denotes the CDF of a random entity $X$ evaluated at a deterministic value of $x$.

If each sensor’s Weibull scale parameter and shape parameter differ from other sensors’ arbitrarily, the corresponding multi-branch SC reception’s outage probability is yet unavailable in closed form in the open literature until now, to the present authors’ best knowledge. The phrase ‘closed form’ here refers to a total absence of any infinite sum and any unresolved integral, even if implicitly inside an ‘incomplete gamma function’, a ‘Marcum Q-function’, or their like.

Available from the open literature on SC reception’s outage probability under Weibull fading.

(a) ‘Closed-form’ $P_{\text{out}}(r_{th})$ expressions are also available for independent branches in [18–20]. These are for only the special case where each sensor’s Weibull fading is independent statistically from all other sensors’ fading, whereby (2) may be simplified to $P_{\text{out}}(r_{th}) = \prod_{i=1}^L \text{Prob}(\gamma_i \leq r_{th}) = \prod_{i=1}^L F_{\gamma_i}(r_{th})$. This is inapplicable to any general scenario where fading is correlated across various sensors, where $P_{\text{out}}(r_{th})$ must be derived from the $L$-variate joint distribution $f_{\gamma_1, \gamma_2, \ldots, \gamma_L}(r_1, r_2, \ldots, r_L)$, instead from the $L$ number of univariate distributions \( \{f_{\gamma_i}(r_i), \forall \ell \} \) separately.

(b) A ‘closed-form’ $P_{\text{out}}(r_{th})$ expression is available for exactly two correlated sensors under the restriction of $\beta_1 = \beta_2$. The expression is equation (6) of [21]. This two-branch expression cannot be readily generalised to more than two branches, even though this two-branch limitation could be constructive for practical diversity-reception systems with three or more branches.

(c) For multi-branch (i.e. $L \geq 3$) correlated fading, only open-form $P_{\text{out}}(r_{th})$ expressions are available [17, 22–24]. These open-form expressions are very complicated, as explained one-by-one below.

(c-1) From [17] (which requires $\Omega_i = \gamma_i, \forall \ell$): $P_{\text{out}}$ is as (3), where $\gamma(x, y) = \int_0^x \int_0^y \, e^{-d} \, dt$ denotes the ‘incomplete gamma function’ and $p$ symbolises the cross-dependence parameter.

(c-2) From [22] (which requires $\beta_i = \beta$, $\Omega_i = \gamma$, $\forall \ell$): $P_{\text{out}}$ is as (4), where the $i$-th entry of the cross-dependence matrix $P$ equals $p_{i,i}$.

(c-3) From [23]: The exact expression would involve an infinite-term summation of summands, each containing the product of three ‘incomplete gamma functions’, each of which involves an unsolved integral. The exact expression would be too complicated to be stated within the limited space allowed for this paper.

(c-4) From [24] (which requires $\beta_i = \beta$, $\gamma_i = \gamma$, $\forall \ell$): $P_{\text{out}}$ is as (5), where $Q$ is the ‘first-order Marcum Q-function’. The above expressions are trail-blazing in their derivations, but these expressions hardly reveal much qualitative insight on how $P_{\text{out}}(r_{th})$ would depend on the model parameters.

From (a) to (c) above, it may be concluded that Weibull-faded multi-branch SC $P_{\text{out}}(r_{th})$ is available up to now in simple closed forms, only for the simplistic degenerate case of statistically independent fading across different sensors, but not for the general case of correlated fading.

1.3 Paper’s contribution and organisation

This paper will propose a new approach that can do away with all aforementioned restrictions in modelling, such that the sensors Weibull-faded envelopes may be cross-correlated, yet each sensor may have its unique Weibull parametric values. This paper pioneers the use of a mathematical paradigm called ‘copula’, to flexibly inter-relate two or more differently distributed univariate scalars, into one joint multivariate random vector. This copula-based approach can lead to simple closed-form expressions for the outage probability of multi-branch ‘selective combining’ reception. [This copula-based approach is potentially applicable to any set of univariate probability distributions pre-selected to model the different fading at different sensors, while satisfying any pre-selected statistical cross-dependence among these different sensors.] Specifically, this paper will demonstrate this new
2 Multivariate Weibull distribution based on the ‘survival-Gumbel’ copula model

Section 2.1 will introduce the probability concept of a ‘copula’ function, in particular. Section 2.2 will define the survival-Gumbel copula function, in particular. Section 2.3 will construct a multivariate Weibull distribution using the survival-Gumbel copula function just introduced. Using this multivariate Weibull distribution, Section 3 will derive $P_{out}(r_0)$ in a closed form, with no unsolved integrals and no infinite summation. Section 4 will present Monte Carlo simulations to verify Section 3’s analytically derived $P_{out}(r_0)$. Section 5 will conclude this entire paper.

2.1 Concept of a copula model

A ‘copula’ [25] can flexibly ‘relate’ different univariate distributions, fashioning them into one multivariate distribution. The mathematical details are presented below.

Given the univariate cumulative distributions

$$F_{X_1}(x), \forall \ell = 1, \ldots, L,$$

each with a domain $R \subseteq \{\infty, -\infty\}$, but otherwise (possibly) different, Sklar’s theorem [25] stipulates the existence of a copula $C(\cdot, \ldots, \cdot)$ with domain $[0, 1]^L$, such that

$$C(F_{X_1}(x_1), \ldots, F_{X_L}(x_2)) = F_{X_1,\ldots,X_L}(x_1, \ldots, x_L)$$

is an L-variate cumulative distribution with domain $(R \cup \{\infty, -\infty\})^L$, satisfying $F_{X_1}(x) = F_{X_1,\ldots,X_L}(\infty, \ldots, \infty, x, \infty, \ldots, \infty)$.

The corresponding multivariate probability density function (6) may be obtained by the Chain Rule.

$$f_{1,2,\ldots,L}(x_1, x_2, \ldots, x_L) = \frac{\partial^L C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_L}(x_L))}{\partial x_1 \partial x_2 \ldots \partial x_L}$$

$$= \frac{\partial^2 C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_L}(x_L))}{\partial F_{X_1}(x_1) \partial F_{X_2}(x_2) \ldots \partial F_{X_L}(x_L)} \frac{df_{X_1}(x_1)}{dx_1} \frac{df_{X_2}(x_2)}{dx_2} \ldots \frac{df_{X_L}(x_L)}{dx_L}$$

$$= \frac{\partial^2 C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_L}(x_L))}{\partial F_{X_1}(x_1) \partial F_{X_2}(x_2) \ldots \partial F_{X_L}(x_L)} f_{1}(x_1) f_{2}(x_2) \ldots f_{L}(x_L)$$

(6)

In the bivariate case of $L = 2$, (6) degenerates to (7).

$$f_{1,2}(x_1, x_2) = \frac{\partial^2 C(F_{X_1}(x_1), F_{X_2}(x_2))}{\partial F_{X_1}(x_1) \partial F_{X_2}(x_2)} f_{1}(x_1) f_{2}(x_2)$$

(7)

2.2 Survival-Gumbel copula function

For the $L$ number of random scalars $X_1, X_2, \ldots, X_L$, the multivariate survival-Gumbel copula function [25] equals (8), where $\theta \geq 1$. If and only if the $L$ random variables are statistically independent, $\theta = 1$.

$$C_{SG}(x_1, x_2, \ldots, x_L; \theta) = \sum_{\ell = 1}^{L} x_{\ell} - (L - 1)$$

$$+ \sum_{\ell = 2}^{L} \left[ (-1)^{\ell} \sum_{1 \leq i < \cdots < k \leq \ell} \exp \left\{ - \sum_{k=1}^{\ell} \left( -\log(1 - x_k) \right)^{\theta} \right\} \right]$$

(8)

For the special case of $L = 4$, the above (8) degenerates to (9). For the trivariate case at $L = 3$, the above (8) degenerates to (10). For the bivariate case at $L = 2$, the above (8) degenerates to (11).

$$C_{SG}(x_1, x_2, x_3; \theta) = x_1 + x_2 + x_3 + x_4 - 3$$

$$+ \exp \left\{ -\log(1 - x_1) + \log(1 - x_2) \right\}$$

$$+ \exp \left\{ -\log(1 - x_1) + \log(1 - x_3) \right\}$$

$$+ \exp \left\{ -\log(1 - x_1) + \log(1 - x_4) \right\}$$

$$+ \exp \left\{ -\log(1 - x_2) + \log(1 - x_3) \right\}$$

$$+ \exp \left\{ -\log(1 - x_2) + \log(1 - x_4) \right\}$$

$$+ \exp \left\{ -\log(1 - x_3) + \log(1 - x_4) \right\}$$

(9)

$$C_{SG}(x_1, x_2, x_3; \theta) = x_1 + x_2 + x_3 - 2 + \exp \left\{ -\log(1 - x_1) \right\}$$

$$+ \exp \left\{ -\log(1 - x_1) \right\}$$

$$+ \exp \left\{ -\log(1 - x_2) \right\}$$

$$+ \exp \left\{ -\log(1 - x_3) \right\}$$

$$+ \exp \left\{ -\log(1 - x_4) \right\}$$

$$+ \exp \left\{ -\log(1 - x_5) \right\}$$

(10)
\[ C_{\text{tot}}(x_1, x_2; \theta) = x_1 + x_2 - 1 + \exp\left[-\left((-\log(1-x_1))^{\beta} + (-\log(1-x_2))^{\beta}\right)^{1/\beta}\right] \]  

(11)

2.3 Multivariate Weibull distribution with the survival-Gumbel copula function

Applying the multivariate copula function of (8) to univariate Weibull cumulative distributions with arbitrary \(\{\beta_i, \Omega_i, \forall i\}\), a copula-based multivariate Weibull cumulative distribution may be obtained as (12).

\[ F_{1, \ldots, L}(z_1, \ldots, z_L) = 1 - \sum_{i=1}^{L} \exp\left(-\frac{z_i}{\Omega_i}\right) + \sum_{i=1}^{L} \left(\prod_{j=1, j\neq i}^{L} \exp\left(-\frac{z_j}{\Omega_j}\right) - \exp\left(-\frac{z_i}{\Omega_i}\right) \right)^{\beta_i - 1} \]  

(12)

[Equation (12) has assumed the same \(\theta\) across all pairs of branches. This assumption is a form of ‘constant correlation’. ‘Constant’ correlation has also been assumed in [17, 24, 26–29], among other references.]

For \(L = 2\), the above degenerates to (13).

\[ F_{1,2}(c_1, z_2) = 1 + \exp\left(-\frac{\beta_1}{\Omega_1}\right) \exp\left(-\frac{\beta_2}{\Omega_2}\right) + \exp\left(-\frac{\beta_1}{\Omega_1}\right) - \exp\left(-\frac{\beta_1}{\Omega_1}\right) \exp\left(-\frac{\beta_2}{\Omega_2}\right) \]  

(13)

[This degenerate bivariate expression is equivalent to (4) in [30]. If furthermore \(\beta_1 = \beta_2\) in (13), then equation (7) of [21] and equation (3.1) in [31] can be obtained. However, none of these three references shows any awareness of their implicit use of any copula. None of these references offers any multivariate distribution as in (12) above.]

Between any \(r\)th and any \(k\)th marginal univariate distributions in the multivariate distribution of (12), the statistical correlation between \(Z_r\) and \(Z_k\) is controlled by the survival-Gumbel copula’s parameter \(\theta\). The familiar ‘Pearson’s correlation’ is deterministically related to \(\theta\) as (14). Therein, \(\Gamma(t) := \int_0^t x^{t-1}e^{-x}dx\) refers to the gamma function.

\[ \rho_{r,k} = \frac{\Gamma(1 + (2/\beta_1)\Gamma(1 + (1/\beta_1)) - \Gamma(1 + (1/\beta_1))\Gamma(1 + (2/\beta_1))}{\Gamma(1 + (2/\beta_1))\Gamma(1 + (2/\beta_1)) - \Gamma(1 + (1/\beta_1))} \]  

(15)

3 \(P_{\text{out}}\) for ‘SC’ under Weibull distribution

Using this multivariate Weibull distribution constructed in Section 2.3, the outage probability (16) at threshold \(r_n\), under multi-branch SC, may be derived via (2), (8), and (12), where \(c_{r_i} = \frac{1}{\Omega_{r_i}/\Omega_i} \forall i\). If \(c_{r_i} = c_{r_j}\), then the single-branch outage probability would be greater at the \(k\)th branch than at the \(r\)th branch.

\[ P_{\text{out}}(r_n) = 1 - \sum_{i=1}^{L} \exp(-c_{r_i}) + \sum_{i=1}^{L} \left(\prod_{j=1, j\neq i}^{L} \exp(-c_{r_j}) - \exp(-c_{r_i}) \right)^{\beta_i - 1} \]  

(16)

The expression in (16)

(i) involves no infinite-term summation and no unsolved integral, whether explicitly or implicitly inside any ‘incomplete gamma function’ nor any ‘Marcum Q-function’ as in [17, 22–24],

(ii) is applicable to \(L \geq 2\), but not limited to \(L = 2\) as is so limited in equation (6) of [21],

(iii) allows the branches to be flexibly cross-correlated, and

(iv) allows each branch of any univariate Weibull statistics, with no restriction whatever on \(\{\beta_i, \Omega_i, \forall i\}\), unlike the multi-branch \(P_{\text{out}}(r_n)\) expressions in [17, 22–24].

These desirable properties if (i)-(iii) are not all obtainable from any of the \(P_{\text{out}}\) expressions in [17, 21–24] nor anywhere else, to the best knowledge of the present authors.

\[ P_{\text{out}}(r_n) = 1 - c^{-c_1} - c^{-c_2} - c^{-c_3} - c^{-c_4} + \exp(-c_1 + c_2)^{1/\beta_1} + \exp(-c_1 + c_3)^{1/\beta_1} \]

(17)

\[ P_{\text{out}}(r_n) = 1 - c^{-c_1} - c^{-c_2} - c^{-c_3} + \exp(-c_1 + c_2)^{1/\beta_1} \]

(18)

\[ P_{\text{out}}(r_n) = 1 - c^{-c_1} - c^{-c_2} \]  

(19)

For the degenerate case of \(L=4\), the above (16) simplifies to (17).

For the degenerate case of \(L=3\), (16) simplifies further to (18).

Lastly, for the degenerate case of \(L=2\), (16) simplifies to (19). If \(\beta_1 = \beta_2\), the above degenerates to equation (9) in [21], which offers
Fig. 1 Three-branch outage probability $P_{\text{out}}(r_{th})$ derived in (18) for a trivariate cross-correlated Weibull distribution, which is constructed via the trivariate ‘survival-Gumbel’ copula of (10)

- $c_1/c_2 = 0.2$ and $\theta = 1.1$
- $c_1/c_2 = 0.2$ and $\theta = 5$
- $c_1/c_2 = 0.8$ and $\theta = 1.1$
- $c_1/c_2 = 0.8$ and $\theta = 5$

Fig. 2 Two-branch $P_{\text{out}}(r_{th})$ analytically derived in (19) (the line), verified by Monte Carlo simulations (the circles), for the scenarios in Section 4.1. Here, $(\beta_1', \Omega_1') = (1.8, 1)$

- $a \quad (\beta_1', \Omega_1') = (1.8, 1.5)$
- $b \quad (\beta_1', \Omega_1') = (2.4, 1)$
- $c \quad (\beta_1', \Omega_1') = (2.4, 1.5)$
These six cases are:

(a) \((\beta_1', \Omega_1') = (2.0, 1.2), (\beta_1', \Omega_1') = (2, 1.5)\), \((\beta_1', \Omega_1') = (2, 2.5)\)

(b) \((\beta_2', \Omega_2') = (1.8, 1.1), (\beta_2', \Omega_2') = (2.1, 1.1), (\beta_2', \Omega_2') = (2.4, 1.1)\)

(c) \((\beta_3', \Omega_3') = (1.8, 1.2), (\beta_3', \Omega_3') = (2.1, 1.5), (\beta_3', \Omega_3') = (2.4, 2.5)\)

no tri-branch \(P_{\text{out}}\) analysis and no multivariate \(P_{\text{out}}\) analysis, and which shows no awareness of the copula approach.

To illustrate the intuitive reasonableness of these derived expressions. Fig. 1 plots the three-branch \(P_{\text{out}(i)}\) of (18) versus the four ‘effective’ parameters of \(c_1/c_2, c_2/c_3, c_3/\ell\), \(\theta\). There, it may be observed (as expected) that \(P_{\text{out}(i)}\) decreases if

(i) \(c_2 = r_{\text{in}}^{\beta_1'/2}/\Omega_1'\) decreases, \(\forall \ell\),
(ii) \(c_1, c_2, \text{ and } c_3\) become more different from each other, and/or
(iii) as \(\theta\) increases, \(P_{\text{out}(i)}\) also decreases.

4 Monte Carlo verification of \(P_{\text{out}}\) derived in (16)

To buttress the reader’s confidence in the copula-based outage probability derived in (16), this section will verify that expression by Monte Carlo simulations for two different scenarios.

4.1 Two cross-correlated branches, but each with its distinct univariate Weibull distribution

Sensor number 1 has \((\beta_1', \Omega_1') = (1.8, 1.1)\), thereby implying that \(c_1 = r_{\text{in}}^{\beta_1'/2}\). Sensor number 2 is tested for six different cases altogether, presented over Figs. 2a–c. These six cases are

(a) \((\beta_2', \Omega_2') = (1.8, 1.5)\), thereby implying that \(c_2 = (2/3)c_1; \text{ and } \theta = 1.1, 5\), which correspond to \(\rho_{\ell, k} \approx 0.1237, 0.9302\), respectively.

(b) \((\beta_2', \Omega_2') = (2.4, 1.0)\), thereby implying that \(c_2 = c_1^{2/3}; \text{ and } \theta = 1.1, 5\), which correspond to \(\rho_{\ell, k} \approx 0.1287, 0.9303\), respectively.

(c) \((\beta_2', \Omega_2') = (2.4, 1.5)\), thereby implying that \(c_2 = 2/3c_1^{2/3}; \text{ and } \theta = 1.1, 5\), which correspond to \(\rho_{\ell, k} \approx 0.1287, 0.9303\), respectively.

In Fig. 2, the red circles show the ‘sample’ cumulative distribution of \(\gamma_{\text{SC}}\), computed from a Monte Carlo experiment consisting of 100,000 statistically independent samples.

For all these six test cases in Fig. 2, the Monte Carlo result coincides with the theoretical curves based on (19), which represents the two-branched degenerate form of (16).

4.2 Four cross-correlated branches, but each with its distinct univariate Weibull distribution

Sensor number 1 has \((\beta_1', \Omega_1') = (2, 1)\). Sensors numbers 2–4 are tested for six different cases altogether, presented over Figs. 3a–c.

These six cases are:

(i) \((\beta_2', \Omega_2') = (2.0, 1.2), (\beta_2', \Omega_2') = (2.0, 1.5), (\beta_2', \Omega_2') = (2.0, 2.5)\); and \(\theta = 1.1, 5\), which correspond to \(\rho_{\ell, k} \approx 0.13, 0.93\), respectively, \(\forall \ell \neq k\).

(ii) \((\beta_2', \Omega_2') = (1.8, 1.0), (\beta_2', \Omega_2') = (2.1, 1.0), (\beta_2', \Omega_2') = (2.4, 1.0)\); and \(\theta = 1.1, 5\), which correspond to \(\rho_{\ell, k} \approx 0.13, 0.93\), respectively, \(\forall \ell \neq k\).

(iii) \((\beta_2', \Omega_2') = (1.8, 1.2), (\beta_2', \Omega_2') = (2.1, 1.5), (\beta_2', \Omega_2') = (2.4, 2.5)\); and \(\theta = 1.1, 5\), which correspond to \(\rho_{\ell, k} \approx 0.13, 0.93\), respectively, \(\forall \ell \neq k\).

In Fig. 3, the red circles again show the ‘sample’ cumulative distribution of \(\gamma_{\text{SC}}\), computed from a Monte Carlo experiment consisting of 100, 000 statistically independent samples.

For all these six test cases in Fig. 3, the Monte Carlo result coincides with the theoretical curves based on (17), which represents the four-branched degenerate form of (16).

5 Conclusion

The probability concept of a ‘copula’ is introduced here to the channel-modelling literature. This ‘copula’ paradigm facilitates the versatile construction of multivariate Weibull distributions – that allows each marginal univariate distribution’s ‘scale parameter’ and ‘shape parameter’ to take arbitrary values, and that allows these various univariate distributions to be cross-correlated. Along with this modelling adaptability, this ‘copula’ approach facilitates the derivation of simple closed-form expressions of the outage probability of SC reception of multiple cross-correlated branches. To the best knowledge of the present authors: this paper advances the first such closed-form expression in the open literature on multi-branched SC reception’s outage probability under cross-correlated Weibull fading.

6 Acknowledgments

Authors were supported by the National Science Council of the Republic of China (NSC 96-2118-M-230-001) and by the Hong Kong Research Grants Council’s General Research Fund number PolyU-152172/14E.

7 References

where $\forall i$,

\[ E[Z_i] = \Omega_i^{1/\beta} \Gamma \left( 1 + 1/\beta_i \right), \]

\[ \text{Var}(Z_i) = \Omega_i^{2/\beta} \left( \Gamma \left( 1 + 2/\beta_i \right) - \Gamma \left( 1 + 1/\beta_i \right)^2 \right), \]

and $E[Z_eZ_d]$ is as (21).

\[ E[Z_eZ_d] = \int_0^\infty z_e z_d f_{Z_e}(z_e) f_{Z_d}(z_d) dz_e dz_d \]

where $i = \Omega_i^{1/\beta} / \beta_i \theta$ for $i = \epsilon$, $k$, $E[Z_eZ_d]$ becomes (22).

\[ E[Z_eZ_d] = \Omega_i^{1/\beta} \Omega_j^{1/\beta} \int_0^\infty \int_0^\infty \exp \left( -\left( \frac{\beta_i \theta + \beta_j \theta}{\beta_i \beta_j \theta} t \right)^{1/\theta} \right) \theta \theta \left( t \theta \right) \text{d}z_e \text{d}z_d \]

Furthermore, let $v_1 = t_1 \beta_i \theta + t_2 \beta_j \theta$ and $v_2 = t_1 \beta_i \theta / v_1$, $E[Z_eZ_d]$ becomes (23).

\[ E[Z_eZ_d] = \Omega_i^{1/\beta} \Omega_j^{1/\beta} \int_0^\infty \int_0^\infty \exp \left( -t_1 \theta \right) \text{d}z_e \text{d}z_d \]

\[ E[Z_eZ_d] = \frac{\Omega_i^{1/\beta} \Omega_j^{1/\beta}}{\beta_i \beta_j \theta} \int_0^t v_1^{a_1} = 1 \int_0^t v_2^{a_2} = 1 \text{d}v_1 \text{d}v_2 \]

Let $s = v_1^{1/\theta}$, $E[Z_eZ_d]$ becomes (25).

\[ E[Z_eZ_d] = \Omega_i^{1/\beta} \Omega_j^{1/\beta} \int_0^\infty \int_0^\infty \exp \left( -t_1 \theta \right) \text{d}z_e \text{d}z_d \]

8 Appendix

8.1 Appendix 1: to derive (14), which relates $\rho_{e,k}$ to $\theta, \beta_i, \beta_j$

The correlation coefficient, by definition, equals (20),

\[ \rho_{e,k} = \frac{\text{Cov}(Z_e, Z_k)}{\sqrt{\text{Var}(Z_e) \text{Var}(Z_k)}} = \frac{E[Z_eZ_k] - E[Z_e]E[Z_k]}{\sqrt{\text{Var}(Z_e) \text{Var}(Z_k)}} \quad (20) \]
Substitute (25) back into the definition of \( \rho_{\ell,k} \) in (20). That gives the desired result.

\[
\frac{\partial}{\partial (\theta^{-1})} M(\theta^{-1}) =
M(\theta^{-1}) \cdot \left[ \frac{1}{\beta_\ell} \psi\left(1 + \frac{1}{\theta \beta_\ell}\right) + \frac{1}{\beta_k} \psi\left(1 + \frac{1}{\theta \beta_k}\right) \right. \left. - \left( \frac{1}{\beta_\ell} + \frac{1}{\beta_k} \right) \psi\left(1 + \frac{1}{\theta \beta_\ell} + \frac{1}{\theta \beta_k}\right) \right]
\]

(26)

8.2 Appendix 2: to prove that \( \rho_{\ell,k} \in [0, 1] \) monotonically increases with \( \theta \in [1, \infty) \): Recall that \( \theta \) affects \( \rho_{\ell,k} \) only through \( E[X_\ell X_k] \), but not through \( E[X_\ell], E[X_k], \) Var(\( X_\ell \)), and Var(\( X_k \)). Moreover

\[
E[X_\ell X_k] = \Gamma\left(1 + \frac{1}{\beta_\ell} + \frac{1}{\beta_k}\right) \frac{\Gamma\left(1 + \frac{1}{\theta \beta_\ell}\right) \Gamma\left(1 + \frac{1}{\theta \beta_k}\right)}{\Gamma\left(1 + \frac{1}{\theta \beta_\ell} + \frac{1}{\theta \beta_k}\right)}.
\]

Since the first factor, \( \Gamma(1 + 1/\beta_\ell)(1/\beta_k) \), is positive, but independent of \( \theta \), \( E[X_\ell X_k] \) is directly proportional to \( M(\theta^{-1}) \).

Hence, to prove that \( \rho_{\ell,k} \) strictly increases with \( \theta \), it would be sufficient to show that \( (\partial/\partial (\theta^{-1})) M(\theta^{-1}) < 0 \), \( \forall \theta > 0 \) at constant \( \beta_\ell \) and \( \beta_k \).

Toward that end, \( (\partial/\partial (\theta^{-1})) M(\theta^{-1}) \) is as (26), where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) symbolises the ‘digamma function’, which is strictly increasing.

Furthermore, as \( \beta_\ell, \beta_k, \theta > 0 \), it holds that

\[
\frac{1}{\beta_\ell} \left[ \psi\left(1 + \frac{1}{\theta \beta_\ell}\right) - \psi\left(1 + \frac{1}{\theta \beta_\ell} + \frac{1}{\theta \beta_k}\right) \right] < 0,
\]

\[
\frac{1}{\beta_k} \left[ \psi\left(1 + \frac{1}{\theta \beta_k}\right) - \psi\left(1 + \frac{1}{\theta \beta_\ell} + \frac{1}{\theta \beta_k}\right) \right] < 0.
\]

Hence, the square-bracketed term in (26) is negative. As \( M(\theta^{-1}) > 0 \), \( \forall \theta \), it follows that \( (\partial/\partial (\theta^{-1})) M(\theta^{-1}) < 0 \), \( \forall \theta > 0 \).

Therefore, the desired result is obtained.