CRB: Sinusoid-Sources’ Estimation using Collocated Dipoles/Loops

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This work derives new asymptotic Cramér-Rao lower bounds (CRB) for the estimation of multiple pure-tone incident signals’ azimuth-elevation arrival-angles, polarization parameters, frequencies, amplitudes, and temporal phases—based on data collected by spatially collocated but orthogonally oriented dipoles and/or loops. The incident sources are pure-tones at distinct, deterministic but unknown frequencies, in contrast to the case of all incident sources at one common known frequency, as has been investigated in the existing research literature on the CRB for diversely-polarized direction-finding. The derived CRBs are closed-form expressions, explicitly in terms of the signal parameters. The new CRBs presented here reveal how a constituent dipole and/or loop’s presence and orientation may affect estimation precision, thereby offering guidelines to the system engineer on what dipole(s) and/or loop(s) to include or to omit in constructing the electromagnetic vector-sensor.

I. INTRODUCTION

A. Electromagnetic Vector-Sensors for Direction-Finding and Polarization Estimation

A completely polarized transverse electromagnetic wave with unit power, traveling through a homogeneous isotropic medium, is characterized in Cartesian spatial coordinates by the six electromagnetic components [8]:

\[
\begin{bmatrix}
    e_x(\psi, \phi, \gamma, \eta) \\
    e_y(\psi, \phi, \gamma, \eta) \\
    e_z(\psi, \gamma, \eta) \\
    h_x(\psi, \phi, \gamma, \eta) \\
    h_y(\psi, \phi, \gamma, \eta) \\
    h_z(\psi, \gamma)
\end{bmatrix}
\]

where \( e(\psi, \phi, \gamma, \eta) \) represents the 3 \times 1 electric-field vector, \( h(\psi, \phi, \gamma, \eta) \) refers to the 3 \times 1 magnetic-field vector, \( 0 \leq \psi \leq \pi \) denotes the signal’s elevation angle measured from the vertical z-axis, \( 0 \leq \phi < 2\pi \) symbolizes the azimuth angle measured from the positive x-axis, \( 0 \leq \gamma < \pi/2 \) refers to the auxiliary polarization angle, and \( -\pi < \eta < \pi \) represents the polarization phase difference. The numerical superscript in \( \Theta^{(i)}(\psi, \phi) \) refers to the number of component-antennas in the vector-sensor, which is defined below.

A series of recent papers have investigated the estimation accuracy for direction-finding and/or polarization estimation using one electromagnetic vector-sensor, which comprises a spatially collocated but diversely oriented collection of electromagnetically short dipoles and/or magnetically small loops. Four specific electromagnetic vector-sensor configurations are as follows.

\[ \Theta^{(i)}(\psi, \phi) = i \]

1This paper will use the following notation: (\( \cdot \))^T for transposition, (\( \cdot \))^H for the Hermitian operation, diag(\( x_1, \ldots, x_N \)) for an \( N \times N \) diagonal matrix with \( x_1, \ldots, x_N \) as the diagonal elements, Re(\() for a complex entity’s real-value part, \( \otimes \) for the Kronecker product, \( \circ \) for an element-wise (Hadamard) product operator, \( j \) for \( \sqrt{-1} \), \( \mathbf{I}_N \) for an \( N \times N \) identity matrix, vec(\() for stacking a matrix’s columns into one column vector, and \( O(N) \) for an order-of-magnitude same as \( N^i \) where \( i \) is an integer.
A dipole-triad \([32, 38]\) consisting of three collocated but orthogonally oriented electrically short dipoles, has been used in \([1, 2, 24, 25, 27, 37, 43, 44, 48]\), and \([49]\) for multiple-source azimuth-elevation direction-finding, tracking and polarization estimation. The dipole-triad’s 3 × 1 array manifold equals \(\mathbf{a}(\psi, \phi, \gamma, \eta) = \mathbf{e}(\psi, \phi, \gamma, \eta)\).

A loop-triad \([40]\) consisting of three collocated but orthogonally oriented magnetically small loops, has been used for the above purpose in \([37, 44]\), and \([48]\). Such a loop-triad is characterized by the 3 × 1 array manifold

\[
\mathbf{a}(\psi, \phi, \gamma, \eta) = \mathbf{h}(\psi, \phi, \gamma, \eta) = \mathbf{e} \left( \psi, \phi, \gamma - \frac{\pi}{2}, -\eta \right) e^{j\eta}
\]

hence, the loop-triad’s array manifold is a duality to the dipole-triad’s. The dipole-triad’s and the loop-triad’s array manifolds may be expressed compactly as

\[
\mathbf{a}(\psi, \phi, \gamma, \eta) = \begin{bmatrix}
\cos \phi \cos \psi & -\sin \phi \\
\sin \phi \cos \psi & \cos \phi \\
-\sin \psi & 0
\end{bmatrix}
\]

where the “flag” \(d\) equals 1 for a dipole-triad and −1 for a loop-triad.

C. A dipole-triad collocated with a loop-triad has been used, towards those same above objectives, in \([4, 6–8, 14–16, 18–21, 26, 28, 29–31, 33–36, 41, 44–48]\), and \([51]\). Such a 6-component structure has a 6 × 1 array manifold \(\mathbf{a}(\psi, \phi, \gamma, \eta, \theta) = \left[ \mathbf{e}(\psi, \phi, \gamma, \eta) \right]^T\), where the superscript T denotes transposition. Six-component electromagnetic vector-sensors are already commercially available, for example, from Flam and Russell Inc. in Horsham, PA, and from EMC Baden Inc. in Baden, Switzerland.

D. Used in \([22, 48, 50]\) and \([52]\) are two horizontally and orthogonally oriented loops collocated with a vertically oriented dipole, matching the incident electromagnetic field emitted from a vertically polarized dipole, a scenario common in mobile uplink propagation. The resulting 3 × 1 array manifold equals \(\mathbf{a}(\psi, \phi, \gamma, \eta) = \left[ \mathbf{e}_1(\psi, \phi, \gamma, \eta), \mathbf{h}_x(\psi, \phi, \gamma, \eta), \mathbf{h}_y(\psi, \phi, \gamma, \eta) \right]^T\).

All above-listed configurations of the electromagnetic vector-sensor offer the following advantages for direction-finding and polarization estimation.

1. Polarization diversity among the vector-sensor’s constituent dipole(s) and/or loop(s) allows incident sources to be separated on account of their polarization differences in addition to their azimuth/elevation angular differences.

2. The component-antennas’ spatial collocation implies no spatial phase delay in the vector-sensor’s array manifold; hence, a vector-sensor can geolocate multiple near-field wideband sources as well as narrowband sources from the far-field, with no additional signal processing.

3. In a multi-source scenario, each source’s three Cartesian direction-cosine estimates (and thus each source’s azimuth-angle estimate and elevation-angle estimate) are automatically paired without additional signal processing.

B. Prior Cramér-Rao Bound Studies for Electromagnetic Vector-Sensors

Closed-form expressions of the Cramér-Rao bounds\(^2\) have been derived in or used in \([8, 11, 14, 21, 31, 35, 37, 44, 46, 47]\) to lower bound the accuracy in direction-finding and/or polarization estimation with the six-component electromagnetic vector-sensors of case C above, but only when all incident signals share one common carrier-frequency and each signal has a base-band-equivalent representation as a temporally uncorrelated stochastic processes.

The above signal model would be inapplicable to the “Uni-Vector-Sensor ESPIRT” algorithms in \([20]\) and \([36]\). Reference \([20]\) handles incident sources that are pure-tone signals and \([36]\) handles frequency-hop signals, which is instantaneously a pure-tone. Such pure-tone signals need to be modeled as deterministic sinusoids with: 1) deterministic, unknown but distinct frequencies, and 2) deterministic and unknown amplitudes, arrival angles, polarization parameters, and temporal phases.\(^3\)

Moreover, the present analysis produces closed-form Cramér-Rao bound expressions that are explicitly in terms of the signal parameters, unlike the

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\(^2\)Theoretical performance bounds, besides the Cramér-Rao bound, for the vector-sensor have been investigated in \([8]\) and \([31]\).

\(^3\)The case with uniformly distributed uncorrelated temporal-phases would be analogous to the subsequent analysis, but with an additional step of taking the statistical expectation of the to-be-derived Fisher information matrix (FIM) with respect to the temporal phases’ uniform probability distribution. The size of the FIM would become smaller, having omitted all rows and columns referring to the temporal phase.
expressions in [8], [10], [11], [21], [31], [35], [37], [44], [46], [47]. Unlike [14], which offers closed-form explicit Cramér-Rao bound expressions only for the polarization parameters, this present analysis produces such expressions for all signal parameters, including the azimuth-elevation arrival directions, frequency, amplitude, and phase.

Cramér-Rao bound analysis for multiple pure-tone signals is available in [3], [5], [9], [12], [13] and [23] (among others), but not for the present diversely polarized electromagnetic vector sensor array manifolds.

II. DERIVATION OF THE NEW CRAMÉR-RAO BOUNDS

A. Statistical Model of the Data in the Present Parameter-Estimation Problem

With a total of $K$ completely polarized pure-tone sources and with additive zero-mean complex-valued Gaussian noise uncorrelated across time and across the component-antennas,$^5$ the data collected at time $t_n$ from the entire electromagnetic vector sensor equals [20]:

$$
\mathbf{z}(t_n) \triangleq \left[ \mathbf{a}(\psi_1, \phi_1, \gamma_1, \eta_1), \ldots, \mathbf{a}(\psi_K, \phi_k, \gamma_k, \eta_k) \right] \times \begin{bmatrix} s(t_n, f_k) \\ \vdots \quad \vdots \\ s(t_n, f_k) \end{bmatrix} + \begin{bmatrix} n(t_n, f_1) \\ \vdots \quad \vdots \\ n(t_n, f_k) \end{bmatrix}, \quad n(t_n, f_k) \triangleq b_k e^{i (2\pi f_k t_n + \varphi_k)}, \quad k = 1, \ldots, K
$$

(4)

where $b_k$ denotes the $k$th source's unknown deterministic amplitude, $f_k$ symbolizes the $k$th signal's distinct but unknown deterministic frequency, and $\varphi_k$ refers to the $k$th signal's unknown temporal phase. The above model differs from that in [8], [11], [14], and [37], wherein each incident signal is modeled as a temporally uncorrelated random sequence.

The entire data set equals:

$$
\mathbf{Z} \triangleq [\mathbf{z}(t_1) \cdots \mathbf{z}(t_N)]
$$

(5)

where $\{t_1, \ldots, t_N\}$ represents the set of distinct time-sampling instants. Subsequent analysis will assume that $t_{i+1} - t_i = \Delta_t$, $\forall i$. To be estimated are the $7K$ unknown deterministic scalars $\{b_k, f_k, \psi_k, \phi_k, \gamma_k, \eta_k, \varphi_k, \} \quad k = 1, \ldots, K$ from $\mathbf{Z}$. Define $\mathbf{\theta} \triangleq [\theta_1^T, \ldots, \theta_K^T]^T$, where

$$
\theta_k \triangleq [b_k, \psi_k, \phi_k, \gamma_k, \eta_k, \varphi_k, f_k]^T.
$$

(6)

Let $\mathbf{z} = \text{vec}(\mathbf{Z}) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Gamma})$ be an $LN \times 1$ vector consisting of all collected time samples, where $L$ denotes the electromagnetic vector sensor’s number of constituent dipole(s) and/or loop(s).

$$
\mathbf{\mu}(\mathbf{\theta}) \triangleq \sum_{k=1}^K b_k \mathbf{a}(\psi_k, \phi_k, \gamma_k, \eta_k) \otimes \mathbf{s}_k,
$$

where

$$
\mathbf{s}_k \triangleq s(f_k, \varphi_k) = \left[ e^{i [2\pi f_k t_1 + \varphi_k]}, \ldots, e^{i [2\pi f_k (N-N+1) + \varphi_k]} \right]^T.
$$

(7)

The additive noise’s known covariance matrix equals

$$
\mathbf{\Gamma} = \mathbf{I}_N \otimes \mathbf{I}_{N \times N},
$$

with

$$
\begin{cases}
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2) \\
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2)
\end{cases}
$$

for the dipole-triad of case A

$$
\begin{cases}
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2) \\
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2, \sigma_h^2, \sigma_h^2)
\end{cases}
$$

for the loop-triad of case B

$$
\begin{cases}
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2) \\
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2, \sigma_h^2, \sigma_h^2)
\end{cases}
$$

for the 6-component of case C

$$
\begin{cases}
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2) \\
\text{diag}(\sigma_h^2, \sigma_h^2, \sigma_h^2, \sigma_h^2, \sigma_h^2)
\end{cases}
$$

for the dipole-plus-two-loops of case D

(8)

This work will derive new Cramér-Rao bounds for the above data statistical model for each of the four aforementioned antennas-collocation cases.

B. Simplifying the Fisher Information Matrix

$$
\text{CRB}(\mathbf{\theta}) = \mathbf{J}^{-1}(\mathbf{\theta}),
$$

where the $7K \times 7K$ Fisher information matrix (FIM),

$$
\mathbf{J}(\mathbf{\theta}) = 2\text{Re} \left[ \left( \frac{\partial \mathbf{\mu}(\mathbf{\theta})}{\partial \mathbf{\theta}} \right)^H \mathbf{\Gamma}^{-1} \frac{\partial \mathbf{\mu}(\mathbf{\theta})}{\partial \mathbf{\theta}} \right]
$$

(9)

may be represented in a block-matrix form in terms of $K^2$ number of submatrices, each of size $7 \times 7$.

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$^4$The derivation here parallels the approach in [39].

$^5$This noise model is used in [8], [11], [14], and [37].

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where

\[ J_{\Sigma,y} = 2\text{Re} \left[ \left( \frac{\partial \mu(\theta)}{\partial \kappa} \right)^{\text{T}} \Gamma^{-1} \frac{\partial \mu(\theta)}{\partial \ell} \right] = J_{y,\Sigma} \]  

and the 7K columns of the \( LN \times 7K \) matrix \( \partial \mu(\theta)/\partial \theta \) equal:

\[
\begin{align*}
\frac{\partial \mu(\theta)}{\partial b_k} &= a_k \otimes s_k \\
\frac{\partial \mu(\theta)}{\partial \psi_k} &= b_k \frac{\partial a_k}{\partial \psi_k} \otimes s_k = b_k \left( \frac{\partial \Theta_k}{\partial \psi_k} \right) s_k \\
\frac{\partial \mu(\theta)}{\partial \phi_k} &= b_k \frac{\partial a_k}{\partial \phi_k} \otimes s_k = b_k \left( \frac{\partial \Theta_k}{\partial \phi_k} \right) s_k \\
\frac{\partial \mu(\theta)}{\partial \eta_k} &= b_k \frac{\partial a_k}{\partial \eta_k} \otimes s_k = b_k \left( \frac{\partial \Theta_k}{\partial \eta_k} \right) s_k \\
\frac{\partial \mu(\theta)}{\partial \varphi_k} &= j b_k a_k \otimes s_k \\
\frac{\partial \mu(\theta)}{\partial f_k} &= 2\pi \Delta f b_k a_k \otimes s_k
\end{align*}
\]

with \( s_k \) already defined in (7), \( \tilde{s}_k \) def \( j [1 - (N + 1)/2, 2 - (N + 1)/2, \ldots, N - (N + 1)/2]^\dagger \otimes s_k \).

Using the matrix identity \((a \otimes b)(B \otimes C)(e \otimes f) = a^B B^H b^H c^H c f \), the 49 elements of the FIM matrix \( J_{k,\ell} \) are determined and are listed in Table I.

C. Decoupling the \( K^2 \) Asymptotic FIM Submatrices

Table I shows that the asymptotic behaviors of the above-mentioned matrices are influenced by \( s_k^H s_k \), \( \bar{s}_k^H \bar{s}_k \), and \( \bar{s}_k^H \bar{s}_k \). The \( K^2 - K \) off-block-diagonal \( 7 \times 7 \) submatrices (corresponding to the cases involving two different incident sources) are shown below as asymptotically (i.e., as \( N \to \infty \)) negligible relative to the block-diagonal submatrices (corresponding to the cases involving only one incident source).

This implies that each incident source’s parameters’ asymptotic Cramér-Rao bounds may be analyzed without consideration of the other sources’ presence.

Note that \( s_k^H s_k \) and \( \bar{s}_k^H \bar{s}_k \) are \( O(N) \) asymptotically, whereas \( s_k^H \bar{s}_k \) is \( O(N^3) \) asymptotically. Define a \( 7 \times 7 \) normalizing matrix \( D_N = \text{diag}(N^{1/2}, N^{1/2}, N^{1/2}, N^{1/2}, N^{1/2}, N^{1/2}, N^{1/2}) \) and the \( 7K \times 7K \) normalizing matrix \( D_N' = \text{diag}(D_{N,1}, \ldots, D_{N,K}) \). All elements in the matrices \( D_N^{-1} J_{k,\ell} D_N^{-1} \) for \( k = 1, \ldots, K \) are asymptotically \( O(1) \), while the other elements are of the order \( O(1/N) \). Hence, the matrix \( D_N^{-1} J(\theta) \bar{D}_N^{-1} \) is asymptotically block-diagonal.

Moreover, the diagonal of \( D_N^{-1} J_{k,k} D_N^{-1} \) consists of the elements \( N^{-1} J_{b_k b_k} \), plus the blocks \( N^{-1} J_{b_k b_k} \), \( k = 1, \ldots, K \). This above also means that the parameters \( b_k \) and \( f_k \) are each asymptotically decoupled from the 5-tuples \((\psi_k, \phi_k, \eta_k, \varphi_k)\) for \( k = 1, \ldots, K \).

These parameters’ CRBs, defined as proper diagonal elements and diagonal blocks of \( J(\theta) \), approximately equal the inverse of \( J_{b_k b_k}, J_{b_k f_k}, \) and \( J_{f_k f_k} \), respectively.

Furthermore, the signal parameters’ asymptotic Cramér-Rao bounds (for large \( N \) and “well separated” frequencies satisfying \( 2\pi|f_j - f_l| \Delta f \gg 1/N \)) are inversely related to the above elements in Table I. Explicit expressions of these are given for the special cases below.

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\[ ^7 \text{This subsection follows the approach in [39].} \]
TABLE I
Entries in 7 × 7 FIM Matrix \( \mathbf{J}_{k,\ell} \)

<table>
<thead>
<tr>
<th>( b_k )</th>
<th>( \psi_k )</th>
<th>( \phi_k )</th>
<th>( \gamma_k )</th>
<th>( \eta_k )</th>
<th>( \varphi_k )</th>
<th>( f_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger \Gamma_0^{-1} a_i s_l s_l^\dagger \right] )</td>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger b_j a_i^\dagger \Gamma_0^{-1} \frac{\partial a_i^\dagger}{\partial \psi_k} s_l^\dagger \right] )</td>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger b_j a_i^\dagger \Gamma_0^{-1} \frac{\partial a_i^\dagger}{\partial \phi_k} s_l^\dagger \right] )</td>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger b_j a_i^\dagger \Gamma_0^{-1} \frac{\partial a_i^\dagger}{\partial \psi_k} s_l^\dagger \right] )</td>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger b_j a_i^\dagger \Gamma_0^{-1} \frac{\partial a_i^\dagger}{\partial \gamma_k} s_l^\dagger \right] )</td>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger b_j a_i^\dagger \Gamma_0^{-1} \frac{\partial a_i^\dagger}{\partial \eta_k} s_l^\dagger \right] )</td>
<td>( \Re \left[ \frac{2}{\sigma_e^2} a_i^\dagger b_j a_i^\dagger \Gamma_0^{-1} \frac{\partial a_i^\dagger}{\partial \varphi_k} s_l^\dagger \right] )</td>
</tr>
</tbody>
</table>

For an arbitrary collection of collocated dipole(s) and/or loop(s), (11) may be expressed as

\[
J_{\alpha,\gamma_k} = 2\Re \left[ \gamma_k \sum_{i=1}^{L} \left( \frac{\partial \mu_i(\theta)}{\partial \alpha_k^\dagger} \right)^\dagger \Gamma_0^{-1} \frac{\partial \mu_i(\theta)}{\partial \gamma_k^\dagger} \right] = J_{\gamma_k,\alpha_k}
\]

(20)

where \( \mu_i(\theta) = \sum_{i=1}^{K} b_j a_i(\psi_k,\phi_k,\gamma_k,\eta_k) \otimes s_k \), \( a_i(\psi_k,\phi_k,\gamma_k,\eta_k) \) refers to the \( i \)th element in the \( L \times 1 \) vector \( a(\psi_k,\phi_k,\gamma_k,\eta_k) \), and \( \Gamma = \sigma I_{N \times N} \) with \( \sigma = \sigma_e \) for \( i = 1,2,3 \) and \( \sigma = \sigma_h \) for \( i = 4,5,6 \). Hence,

\[
J_{k,k} = \sum_{i=1}^{L} J_{k,k,i}^i \quad (21)
\]

where \( J_{k,k,i}^i \) refers to the entity obtained as in (9) but with \( \mu(\theta) \) substituted by \( \mu_i(\theta) \).

D. Special Cases of a Dipole-Triad or a Loop-Triad

For the special cases of a dipole-triad or a loop-triad (where (3) applies), it is shown below that for \( k = 1,\ldots,K \),

\[
J_{k,k} = \begin{bmatrix}
O(N) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & O(N) & O(N) & 0 & 0 & 0 & 0 \\
0 & O(N) & O(N) & O(N) & O(N) & 0 & 0 \\
0 & 0 & O(N) & O(N) & 0 & 0 & 0 \\
0 & 0 & 0 & O(N) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & O(N) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & O(N) & 0
\end{bmatrix}
\]

(22)
The approximate CRB expression for $f_k$ will be proportional to $N^{-3}$, as generally the case in frequency estimation; and the CRBs for the other six parameters will be proportional to $N^{-1}$.

Define $\sigma = \sigma_f$ for the dipole-triad, or define $\sigma = \sigma_h$ for the loop-triad

$$J_{b_1,a_h} = \frac{2N}{\sigma^2} \text{Re}[a_h^\dagger a_h] = \frac{2N}{\sigma^2} (\text{tr}[g_h])$$

$$J_{\psi_k,v_k} = \frac{2b_k^2N}{\sigma^2} \text{Re} \left[ (g_k^d)^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} g_k^d \right]$$

$$= \frac{2b_k^2N}{\sigma^2} \sin^2(\gamma_k - \pi/4 + d\pi/4)$$

$$J_{\psi_k,v_h} = \frac{2b_k^2N}{\sigma^2} \text{Re} \left[ (g_k^d)^\dagger \begin{bmatrix} \cos(\psi_k) & 0 \\ 0 & 1 \end{bmatrix} g_k^d \right]$$

$$= \frac{2b_k^2N}{\sigma^2} \cos^2(\psi_k) \sin^2(\gamma_k - \pi/4 + d\pi/4) + \cos^2(\gamma_k - \pi/4 + d\pi/4))$$

$$J_{\psi_k,v_h} = \frac{2b_k^2N}{\sigma^2} \sin^2(\gamma_k)$$

$$J_{\psi_k,v_h} = \frac{2b_k^2N}{\sigma^2} \frac{2\pi^2 b_k^2 (N^2 - 1) \Delta^2}{\Delta^2}$$

Moreover, (19) gives $J_{b_1,b_2} = J_{b_1,v_k} = J_{b_1,v_h} = J_{b_2,v_k} = J_{b_2,v_h} = J_{v_k,v_h} = J_{v_k,v_h} = 0$; (71) gives $J_{b_1,v_h} = J_{b_2,v_h} = J_{v_k,v_h} = 0$; (72) gives $J_{b_1,v_h} = (2/\sigma^2) \text{Re} \left[ b_h^\dagger (\partial a_h/\partial \gamma_k) \gamma_k^d a_h^d \right] = 0$; the Appendix’s (88) gives $J_{b_1,v_h} = J_{b_2,v_h} = J_{v_k,v_h} = J_{v_k,v_h} = 0$. Furthermore, what is inside the following entities’ real-value brackets are all purely imaginary-valued: $J_{b_1,v_h} = (2/\sigma^2) \text{Re} \left[ b_h^\dagger (\partial a_h/\partial \gamma_k) \gamma_k^d a_h^d \right] = 0$ according to (68) and (80), $J_{b_1,v_h} = 0$ according to (17), $J_{b_2,v_h} = 0 = (2N b_k^2/\sigma^2) \text{Re} [j \sin(\gamma_k + (d-1)d\pi/4) \sin^2(\gamma_k + (d-1)d\pi/4)]$ according to the Appendix’s (74).

The non-zero entries are derived as follows. Using (83),

$$J_{\psi_k,v_h} = \frac{2N b_k^2}{\sigma^2} \text{Re}[g_k^d]^\dagger \begin{bmatrix} 0 & \sin(\psi_k) \\ 0 & 0 \end{bmatrix} g_k^d$$

$$= \frac{2N b_k^2}{\sigma^2} \sin(\psi_k) \sin\left(\gamma_k + \frac{(d-1)d\pi}{4}\right)$$

$$\times \cos\left(\gamma_k + \frac{(d-1)d\pi}{4}\right) \cos(d\eta_k) = O(N)$$

$$J_{\psi_k,v_h} = \frac{2N b_k^2}{\sigma^2} \cos(\psi_k) \cos(d\eta_k) = O(N)$$

Substituting the above derived expressions into Table I to obtain $J_{b_k,h}$ give for the dipole-triad the following asymptotic Cramér-Rao bounds:

$$\text{CRB}_{b_k} = \frac{\sigma^2}{2N} + O(N^{-2})$$

$$\text{CRB}_{\psi_k} = \frac{\sigma^2}{2N} \frac{b_k}{b_k^2} \left( \sin^2(\gamma_k) \sin^2(\eta_k) \right) + O(N^{-2})$$

$$\text{CRB}_{\psi_k} = \frac{\sigma^2}{2N} \frac{1}{b_k^2} \cos^2(\gamma_k) \sin^2(\eta_k) + O(N^{-2})$$

The loop-triad has these corresponding asymptotic Cramér-Rao bounds:

$$\text{CRB}_{b_k} = \frac{\sigma^2}{2N} + O(N^{-2})$$

$$\text{CRB}_{\psi_k} = \frac{\sigma^2}{2N} \frac{1}{b_k^2} \cos^2(\gamma_k) \sin^2(\eta_k) + O(N^{-2})$$

$$\text{CRB}_{\psi_k} = \frac{\sigma^2}{2N} \frac{1}{b_k^2} \cos^2(\gamma_k) \sin^2(\eta_k) + O(N^{-2})$$

$$\text{CRB}_{\psi_k} = \frac{\sigma^2}{2N} \frac{1}{b_k^2} \cos^2(\gamma_k) \sin^2(\eta_k) + O(N^{-2})$$

$$\text{CRB}_{\psi_k} = \frac{\sigma^2}{2N} \frac{1}{b_k^2} \cos^2(\gamma_k) \sin^2(\eta_k) + O(N^{-2})$$
Fig. 1. \((2b_k^2N/\sigma_e^2)\text{CRB}_{\psi_k}^{(e)}\) versus \(\eta_k\) and \(\gamma_k\).

\[
\text{CRB}_{\psi_k}^{(h)} = \frac{\sigma_e^2}{2N b_k^2} \left\{ 1 + \left[ \frac{\cot(\psi_k) \cot(\eta_k)}{\sin \gamma_k} \right]^2 \right\} + O(N^{-2})
\]

\[
\text{CRB}_{\eta_k}^{(h)} = \frac{\sigma_e^2}{N b_k^2} \left\{ \frac{1}{\sin^2(2\gamma_k)} + \frac{\cot^2(\psi_k)}{\sin^2 \gamma_k} \left[ \frac{1}{\sin^2(2\gamma_k)} - 1 \right] \right\}
\]

\[
+ O(N^{-2})
\]

\[
\text{CRB}_{\gamma_k}^{(h)} = \frac{\sigma_e^2}{2N b_k^2} \frac{1}{\cos^2(\gamma_k) \sin^2(\psi_k)}
\]

\[
+ O(N^{-2}).
\]

Fig. 1 plots \((2b_k^2N/\sigma_e^2)\text{CRB}_{\psi_k}^{(e)}\) of (37) versus \(\eta_k\) and \(\gamma_k\). Fig. 2 plots \((2b_k^2N/\sigma_e^2)\sin^2(\psi_k)\text{CRB}_{\psi_k}^{(e)}\) of (38) versus \(\eta_k\) and \(\gamma_k\). Figs. 1 and 2 clearly reveal their sinusoidal dependence on \(|\eta_k|\) and on \(\gamma_k\). Figs. 3 and 4 plot \((2b_k^2N/\sigma_e^2)\text{CRB}_{\eta_k}^{(e)}\) of (39) versus \(\psi_k\) and \(\gamma_k\), respectively at \(\eta_k = \pi/8\) and \(3\pi/8\). The nearer \(|\eta_k|\) is to \(\pi/2\), the smaller \(\cot^2 \eta_k\) becomes; that reduces the influence of \(\cot \psi_k/\cot \gamma_k\) (relative to 1) on \(\text{CRB}_{\eta_k}^{(e)}\) and flattens the graph. A similar argument could be made for \(\psi_k\) on \(\text{CRB}_{\psi_k}^{(e)}\). Fig. 5 plots \((b_k^2N/2\sigma_e^2)\text{CRB}_{\eta_k}^{(e)}\) of (40) versus \(\psi_k\) and \(\gamma_k\).

Below are some observations on the above-derived asymptotic Cramér-Rao bound expressions.

1) The dipole-triad's and the loop-triad's Cramér-Rao bounds are both independent of the azimuth arrival-angle \(\phi_k\) (which is defined on the x-y plane). This is because both configurations have identical component-antennas symmetrically oriented along the x-axis and the y-axis.

2) For both the dipole-triad and the loop-triad, the frequency's Cramér-Rao bounds are approximately proportional to \(1/1^2\), whereas all other parameters' Cramér-Rao bounds are approximately proportional to \(1/N\).

3) For both the dipole-triad and the loop-triad, all Cramér-Rao bounds are minimized when the polarization phase difference \(\eta_k = \pi/2\) or the elevation angle \(\psi_k = \pi/2\).
Fig. 3. \( (2h^2N/\sigma_x^2)\text{CRB}^{(e)}_{\psi_k} \) versus \( \psi_k \) and \( \gamma_k \), at \( \eta_k = \pi/8 \).

Fig. 4. \( (2h^2N/\sigma_x^2)\text{CRB}^{(e)}_{\psi_k} \) versus \( \psi_k \) and \( \gamma_k \), at \( \eta_k = 3\pi/8 \).

Fig. 5. \( (\theta^2N/2\sigma_x^2)\text{CRB}^{(e)}_{\psi_k} \) versus \( \psi_k \) and \( \gamma_k \).
4) Due to the duality between the dipole-triad’s and the loop-triad’s array manifolds in (2), the dipole-triad’s Cramér-Rao bounds in (35) to (40) may be transformed into the loop-triad’s corresponding Cramér-Rao bounds in (42) to (47) by substituting \( \cos^2 \gamma_k \) with \( \sin^2 \gamma_k \), \( \sin^2 \gamma_k \) with \( \cos^2 \gamma_k \), and \( \sigma_e \) with \( \sigma_h \), except for \( \phi_k \). Hence, the loop-triad’s counterparts to the dipole-triad’s Figs. 1 to 5 will not be shown.

5) These identities hold:

\[
\frac{\text{CRB}^{(e)}_{\phi_k}}{\text{CRB}^{(h)}_{\phi_k}} \left( \frac{\sigma_k}{\sigma_e} \right)^2 = \frac{\text{CRB}^{(h)}_{\phi_k}}{\text{CRB}^{(e)}_{\phi_k}} \left( \frac{\sigma_k}{\sigma_h} \right)^2 = \sin^2 \psi_k.
\]

Fig. 6 shows how the incident sources’ frequency separation affects their dipole-triad direction-cosine estimates’ nonasymptotic Cramér-Rao bound at different signal-to-noise ratios (SNR), in a two-source scenario when the number \( N \) of time-samples is finite. Note that the asymptotic (as \( N \rightarrow \infty \)) Cramér-Rao bounds in (35) to (41) are inapplicable for this finite-sample scenario. The nonasymptotic Cramér-Rao bound here is computed by inverting the exact information matrix in (9). This computation depends on the temporal phase; hence, each data-point here represents the average of 200 statistically independent trials with the temporal phase uniformly distributed in \([0, 2\pi]\). A higher SNR improves the direction-cosines’ estimates’ accuracy but does not narrow the frequency-separation requirement for successful resolution of the direction-cosines. Similar trends hold for the three other vector-sensor configurations.

E. Special Case of a Six-Component Electromagnetic Vector-Sensor

From (20), the FIM equals:

\[
J_{k_k} = J^{(e)}_{k_k} + J^{(h)}_{k_k}
\]

where \( J^{(e)}_{k_k} \) denotes the dipole-triad’s FIM for the \( k \)th source; and \( J^{(h)}_{k_k} \) denotes the loop-triad’s FIM for the \( k \)th source. This leads to the following Cramér-Rao bounds:

\[
\text{CRB}^{(eh)}_{\theta_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{\sigma^2_e + \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\psi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2 (N^2 - 1) \pi^2 (\sigma^2_e + \sigma_h^2) \Delta h^2} + O(N^{-4})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]

\[
\text{CRB}^{(eh)}_{\phi_k} = \frac{\sigma^2 \sigma_e^2}{2N} \frac{1}{b_k^2} \frac{\sigma^2 \sin^2 \gamma_k + \sigma_h^2 \cos^2 \gamma_k}{\lambda^2 (\sigma^2_e - \sigma^2_h) \sin^2 \eta_k \cos^2 \gamma_k \sin^2 \gamma_k + \sigma_e^2 \sigma_h^2} + O(N^{-2})
\]
Fig. 6. The root-mean square (RMS) nonasymptotic (for a finite number of data samples) Cramér-Rao bound of the direction-cosine estimates using dipole-triad, versus two incident sources’ digital-frequency separation. The two closely spaced sources have \( \{\psi_1, \psi_2\} = \{75^\circ, 80^\circ\}, \{\phi_1, \phi_2\} = \{35^\circ, 30^\circ\}, \{\gamma_1, \gamma_2\} = \{45^\circ, 40^\circ\}, \{\eta_1, \eta_2\} = \{25^\circ, 20^\circ\}, \) \( b_1 = b_2 = 1.\) Moreover, \( \Delta_T = 0.1 \) and 100 snapshots with uniform sampling frequency in each of the 200 experiments with independently distributed phases.

Fig. 7. \( (2b_k^2 N/\sigma_k^2) \text{CRB}_{\psi_k} \) versus \( \eta_k \) and \( \gamma_k, \) at \( (\sigma_e/\sigma_h) = \sqrt{2}. \)

Fig. 8. \( (2b_k^2 N/\sigma_k^2) \text{CRB}_{\psi_k} \) versus \( \eta_k \) and \( \gamma_k, \) at \( (\sigma_e/\sigma_h) = 2. \)
Figs. 7 and 8 plot \((2b_k^2N/\sigma_\epsilon^2)\sin^2(\psi_k)\text{CRB}^{(\epsilon)}_{\psi_k}\) of (52) versus \(\eta_k\) and \(\gamma_k\), respectively at \((\sigma_\epsilon/\sigma_h) = \sqrt{2}\) and 2. Figs. 9 and 10 plot \((2b_k^2N/\sigma_\epsilon^2)\sin^2(\psi_k)\text{CRB}^{(h)}_{\psi_k}\) of (53) versus \(\eta_k\) and \(\gamma_k\), respectively at \((\sigma_\epsilon/\sigma_h) = \sqrt{2}\) and 2. From these four figures, we observe that the closer \(\sigma_\epsilon\) and \(\sigma_h\) are, the flatter the surface. In the limiting case where \(\sigma_\epsilon = \sigma_h\), both \(\text{CRB}^{(\epsilon)}_{\psi_k}\) and \(\sin^2(\psi_k)\text{CRB}^{(h)}_{\psi_k}\) become independent of all direction-of-arrival parameters and polarization parameters.

Observations (1–3) for the dipole-triad or the loop-triad in Section IID also hold for the present six-component vector-sensor configuration. Additionally, the following apply specifically to the six-component vector-sensor’s asymptotic Cramér-Rao bounds.

1) For \(\sigma_\epsilon = \sigma_h\), \(\text{CRB}^{(\epsilon)}_{\psi_k} = \text{CRB}^{(h)}_{\psi_k} \sin^2 \psi_k\), \(\text{CRB}^{(\epsilon)}_{\theta_k} = \frac{1}{2} \text{CRB}^{(\epsilon)}_{\psi_k}\), \(\text{CRB}^{(h)}_{\theta_k} = \frac{1}{2} \text{CRB}^{(h)}_{\psi_k}\), and \(\text{CRB}^{(\epsilon)}_{\phi_k} = \frac{1}{2} \text{CRB}^{(h)}_{\phi_k}\).

2) Because

\[
\frac{1}{\text{CRB}^{(\epsilon)}_{b_k}} + \frac{1}{\text{CRB}^{(h)}_{b_k}} = \frac{1}{\text{CRB}^{(\epsilon)}_{b_k}} \frac{1}{\text{CRB}^{(h)}_{h_k}} + \frac{1}{\text{CRB}^{(h)}_{h_k}}
\]

the six-component vector-sensor’s asymptotic Cramér-Rao bounds for \(b_k\) and \(b_k\) are lower than the corresponding bounds for the dipole-triad and the loop-triad. This is expected, as there exist more component-antennas in the six-component vector-sensor.

F. Special Case of One Vertical Dipole Plus Two Horizontal Loops

For the configuration of case D, (20) gives the FIM,

\[
J_{k,k} = J^{(\epsilon)}_{k,k} + J^{(h)}_{k,k} + J^{(b)}_{k,k}
\]
where \( J^{(k)}_{kk} \) denotes the vertical dipole’s FIM for the \( k \)th source; \( J^{(b)}_{kk} \) and \( J^{(b)}_{kk} \) symbolize, respectively, the \( x \)-axis and \( y \)-axis loop’s FIM for the \( k \)th source. This leads to the following Cramér-Rao bounds:

\[
\text{CRB}_{\theta_k}^{(d)} = \frac{(\sigma^2 + \sigma^2_h)(\sin^2 \psi_k)(\cos^2 \gamma_k \sin^2 \eta_k + \cos^2 \eta_k) + \sigma^2_k (\cos^2 \psi_k)(\sin^2 \gamma_k \sin^2 \eta_k - \sin^2 \psi_k)}{2N \cos^2 \psi_k \sin^2 \eta_k \sin^2 \gamma_k} + O(N^{-2})
\]

\[
\text{CRB}_{\theta_k}^{(l)} = \frac{3\sigma^2 \sigma^2_h}{2N(N^2 - 1)^2 b^2 \Delta^2 ((\sigma^2_k \sin^2 \psi + \sigma^2_k \sin^2 \gamma_k) + \sigma^2_k \cos^2 \psi_k \cos^2 \gamma_k)} + O(N^{-4})
\]

\[
\text{CRB}_{\theta_k}^{(e)} = \frac{1}{2b^2 N} \left( \sigma^2 + \sigma^2_h \right) (\sin^2 \psi_k) \left( (\cos^2 \eta_k - \sigma^2_k \cos^2 \psi_k \sin^2 \psi_k) (\sigma^2_k + \sigma^2_h)^2 \right) (\sin^2 \eta_k) + O(N^{-2})
\]

Observations (1–3) for the dipole-triad or the loop-triad in Section IID also hold for the present one-dipole-plus-two-loops’ asymptotic Cramér-Rao bounds. Additionally, the following apply specifically to this one-dipole-plus-two-loops configuration:

\[
\text{CRB}_{\theta_k}^{(e2)} = \frac{1}{2b^2 N} \left( \sigma^2 + \sigma^2_h \right) \left( \gamma_k (\sin^2 \psi_k) (\cos^2 \eta_k + \cos^2 \gamma_k \sin^2 \eta_k \sin^2 \psi_k) (\sigma^2_k + \sigma^2_h) \right) + O(N^{-2})
\]

\[
\text{CRB}_{\theta_k}^{(e3)} = \frac{1}{2b^2 N} \left( \sigma^2 + \sigma^2_h \right) \left( \gamma_k (\sin^2 \psi_k) (\cos^2 \eta_k + \cos^2 \gamma_k \sin^2 \eta_k \sin^2 \psi_k) (\sigma^2_k + \sigma^2_h) \right) + O(N^{-2})
\]

\[
\text{CRB}_{\theta_k}^{(e4)} = \frac{1}{2b^2 N} \left( \sigma^2 + \sigma^2_h \right) \left( \gamma_k (\sin^2 \psi_k) (\cos^2 \eta_k + \cos^2 \gamma_k \sin^2 \eta_k \sin^2 \psi_k) (\sigma^2_k + \sigma^2_h) \right) + O(N^{-2})
\]

\[
\text{CRB}_{\theta_k}^{(e5)} = \frac{1}{2b^2 N} \left( \sigma^2 + \sigma^2_h \right) \left( \gamma_k (\sin^2 \psi_k) (\cos^2 \eta_k + \cos^2 \gamma_k \sin^2 \eta_k \sin^2 \psi_k) (\sigma^2_k + \sigma^2_h) \right) + O(N^{-2})
\]

AU-YEUNG & WONG: CRB: SINUSOID-SOURCES’ ESTIMATION USING COLLOCATED DIPOLES/LOOPS
1) \[ \frac{\text{CRB}_{\psi_k}^{(e1h2)}}{\text{CRB}_{\psi_k}^{(e)}} = \frac{\text{CRB}_{\phi_k}^{(e1h2)}}{\text{CRB}_{\phi_k}^{(e)}} = \frac{\sigma_k^2 \sin^2 \psi_k + \sigma_k^2}{\cos^2 \psi_k}. \]

2) \[ \text{CRB}_{\psi_k}^{(e1h2)} \text{ of (59) converges to } \text{CRB}_{\phi_k}^{(h)} \text{ (43) as } \psi_k \to 0. \text{ That is, when the incident transverse electromagnetic signal impinges from the vertical axis (and thus cannot have any electric-field or magnetic-field component along the z-axis), the presence or absence of any z-axis dipole or loop would make no difference.} \]

3) Comparing among \[ \text{CRB}_{x}^{(e)}, \text{CRB}_{x}^{(h)}, \text{CRB}_{x}^{(eh)}, \text{CRB}_{x}^{(e1h2)} \text{ for any to-be-estimated parameter } x \in \mathcal{P}_k = \{b_k, f_k, \psi_k, \phi_k, \gamma_k, \eta_k, \varphi_k \}, \text{CRB}_{x}^{(eh)} \text{ and } \text{CRB}_{x}^{(h)} \text{ depend on the same subset of } \mathcal{P}_k, \text{ whereas } \text{CRB}_{x}^{(eh)} \text{ and } \text{CRB}_{x}^{(e1h2)} \text{ generally depend on more members of } \mathcal{P}_k. \]

III. CONCLUSION

In this paper the asymptotic Cramér-Rao bounds are derived in closed form, for direction-finding and polarization estimation of uncorrelated pure sinusoidal point sources under four different configurations: a dipole-triad, a loop-triad, a collocated dipole, and loop triad, as well as two horizontally oriented loops collocated with one vertically oriented dipole. These closed-form explicit expressions of the Cramér-Rao bounds reveal that numerous qualitative rule-of-thumb insights on various constructions of the vector-sensor would affect various to-be-estimated parameters’ estimation accuracy. These insights can help the system engineer choose among various possible constructions of the electromagnetic vector-sensor. That is, the system engineer now has more rule-of-thumb guidelines on which dipole or loop to include or to omit in the vector sensor.

IV. APPENDICES

A. Formulas used in Deriving the Fisher Information Matrices

Towards deriving the explicit closed-form expressions of the asymptotic Cramér-Rao bounds of \[ b_k, \psi_k, \phi_k, \gamma_k, \eta_k, \varphi_k, \text{ and } f_k, \text{ note that:} \]

\[ (g_k^{(d)})^H g_k^{(d)} = 1 \] \hspace{1cm} (65)

\[ (g_k^{(d)})^H \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} g_k^{(d)} = A \sin^2 (\gamma_k - \pi/4 + d\pi/4) + B \cos^2 (\gamma_k - \pi/4 + d\pi/4) \] \hspace{1cm} (66)

\[ (g_k^{(d)})^H \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} g_k^{(d)} = A \sin (\gamma_k - \pi/4 + d\pi/4) \times \cos (\gamma_k - \pi/4 + d\pi/4)e^{-j\eta_k} + B \sin (\gamma_k - \pi/4 + d\pi/4) \times \cos (\gamma_k - \pi/4 + d\pi/4)e^{j\eta_k} \] \hspace{1cm} (67)

for any scalar constants \( A \) and \( B \). If \( A = -B \), then the right-hand side of (67) becomes

\[ 2Bj \sin (\gamma_k - \pi/4 + d\pi/4) \cos (\gamma_k - \pi/4 + d\pi/4) \sin (\eta_k d). \] \hspace{1cm} (68)

Note that (66) and (68) are real-valued and imaginary-valued, respectively.

Furthermore,

\[ \frac{\partial (g_k^{(d)})}{\partial \gamma_k} = \frac{\cos (\gamma_k - \pi/4 + d\pi/4)e^{j(1+\eta_k)/2} - j \sin (\gamma_k - \pi/4 + d\pi/4)e^{j(1-\eta_k)/2}}{1+2} \] \hspace{1cm} (69)

\[ \frac{\partial (g_k^{(d)})}{\partial \eta_k} = \frac{j \frac{1-d}{2} \sin (\gamma_k - \pi/4 + d\pi/4)e^{j(1+\eta_k)/2} + j \frac{1-d}{2} \cos^2 (\gamma_k - \pi/4 + d\pi/4)}{1+2} \] \hspace{1cm} (70)

\[ \frac{\partial (g_k^{(d)})}{\partial \gamma_k} = 0 \] \hspace{1cm} (71)

\[ \frac{\partial (g_k^{(d)})}{\partial \eta_k} = j \frac{1+d}{2} \sin (\gamma_k - \pi/4 + d\pi/4) + j \frac{1+d}{2} \cos^2 (\gamma_k - \pi/4 + d\pi/4) \] \hspace{1cm} (72)

\[ \left( \frac{\partial (g_k^{(d)})}{\partial \gamma_k} \right)^H \frac{\partial (g_k^{(d)})}{\partial \eta_k} = 1 \] \hspace{1cm} (73)

\[ \left( \frac{\partial (g_k^{(d)})}{\partial \eta_k} \right)^H \frac{\partial (g_k^{(d)})}{\partial \eta_k} = j d \cos (\gamma_k - \pi/4 + d\pi/4) \times \sin (\gamma_k - \pi/4 + d\pi/4) \] \hspace{1cm} (74)

\[ \left( \frac{\partial (g_k^{(d)})}{\partial \eta_k} \right)^H \frac{\partial (g_k^{(d)})}{\partial \eta_k} = \sin^2 \gamma_k. \] \hspace{1cm} (75)

Recall that \( \Theta^{(3)}(\psi_k, \phi_k) = \Theta^{(3)} \) is common to both the dipole-triad and the loop-triad,

\[ (\Theta_k^{(3)})^H \Theta_k^{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] \hspace{1cm} (76)

\[ \frac{\partial \Theta_k^{(3)}}{\partial \psi_k} = \begin{bmatrix} -\cos \phi_k \sin \psi_k \\ -\sin \phi_k \sin \psi_k \\ -\cos \psi_k \end{bmatrix} \hspace{1cm} (77)

\[ \frac{\partial \Theta_k^{(3)}}{\partial \phi_k} = \begin{bmatrix} -\sin \phi_k \cos \psi_k - \cos \phi_k \\ \cos \phi_k \cos \psi_k - \sin \phi_k \\ 0 \end{bmatrix} \hspace{1cm} (78)

\[ (\Theta_k^{(3)})^H \frac{\partial \Theta_k^{(3)}}{\partial \psi_k} = 0 \] \hspace{1cm} (79)
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