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Generation of moment invariants and their uses for character recognition

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Abstract

An automatic approach for the generation of moment invariants is presented. By making use of this approach a complete set of moment invariants is generated, many of which have never been reported in the literature. This approach will guarantee not to miss any possible expression that may be useful for pattern recognition. Consequently, the generated sets of moment invariants, including higher-order moments, have been used for the recognition of English alphabets. We also include the results of our study on selecting the best set of weighting factors for invariant moments, such that we have improved the recognition rates to over 98.8%.

Keywords: Moment invariants; Euclidean classification; Weighting factor

1. Introduction

Moment invariants were first introduced by Hu (1961, 1962) and a large number of papers (Wong and Hall, 1978; Dudani et al., 1977; Hais, 1981; Belkasim et al., 1991) that have significant contributions to the application of the subject appeared afterward. But only moment invariants of third order or below third order were described in these papers. Some higher-order moment invariants (up to seventh order) can be found in (Belkasim et al., 1991). However, only a small subset of moment invariants was listed in the papers mentioned. In principle, there is an infinitely large number of moment invariants and most of them have not been reported in the literature. Manual calculation and brute force techniques have been used for the generation of moment invariants, which may be considered as the reason why other moment invariants have not been reported and used in image processing. In this paper we develop an automatic approach for the generation of the moment invariants and give the best set of weighting factors for character recognition.

Let us recall the definition of two-dimensional moments of order $(p+q)$ of a density distribution function $f(x, y)$ as follows:

$$m_{pq}=\iint x^p y^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y\,. \tag{1}$$

The moments that have the property of translation invariance are called central moments and are defined as

$$\mu_{pq}=\iint (x-\bar{x})^p (y-\bar{y})^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{2}$$

where $\bar{x}=m_{10}/m_{00}$, $\bar{y}=m_{01}/m_{00}$ are the coordinates of the centroid.

The normalized central moments, which are invar-

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iant to the scale of the image, can be defined as

$$\eta_{pq} = \mu_{pq} / \mu_{00}^{\gamma} \tag{3}$$

where $\gamma = (p+q)/2+1$.

Under the following orthogonal transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

and by the method of algebraic invariant as described in (Hu, 1962), the following $p+1$ monomials for moments up to order p can be obtained:

$$I'_{p0} = e^{ip\theta} I_{p0}; \quad I'_{p-1,1} = e^{i(p-2)\theta} I_{p-1,1}; \quad \dots ;$$
$$I'_{1,p-1} = e^{-i(p-2)\theta} I_{1,p-1}; \quad I'_{0p} = e^{-ip\theta} I_{0p} . \tag{5}$$

The general form of the coefficients of the invariant identity in Eq. (5) can be written as follows:

$$I'_{pq} = I_{pq} \exp[i(p-q)\theta] \tag{6}$$

where I'_{pq} is the coefficient in the rotated domain, I_{pq} is the coefficient in the original domain, $p+q$ is the order of the moment.

2. Automatic generation of moment invariants

The moment invariants can be generated by eliminating the rotational dependent term θ in Eq. (6). Let us give some simple examples to illustrate the idea.

For second-order moments, the moment invariants are: I_{11}, $I_{20} \cdot I_{02}$.

For third-order moments, some of the moment invariants are: $I_{30} \cdot I_{03}$, $I_{21} \cdot I_{12}$, $I_{21}^2 \cdot I_{02}$.

From the examples given above, the moment invariants can be obtained by multiplying a suitable combination of terms in order to eliminate the exponential factor in Eq. (6). For some simpler cases of lower-order moments, these can be done by manual calculation. However, it is impossible to try out all combinations of terms from different order of moments by hand calculation. In this section, we make use of a basic idea of number theory to find out all possible combinations of moment invariants. Our algorithm for the generation is given in the following five steps.

Step 1. Generate a link list which contains the difference of p and q for each term defined in Eq. (6) for a specified order of moments. Example: let us use moments up to third order to illustrate our idea in this step, and for the rest of the steps, i.e.

$$I_{20} = 2, \quad I_{11} = 0, \quad I_{02} = -2 ,$$
$$I_{30} = 3, \quad I_{21} = 1, \quad I_{12} = -1, \quad I_{03} = -3 .$$

Step 2. Assign a variable X_i associated with each entry of the link list described in Step 1 in order to form a new polynomial as shown below:

$$X_0(2) + X_1(0) + X_2(-2) + X_3(3) + X_4(1)$$
$$+ X_5(-1) + X_6(-3) = 0 . \tag{7}$$

Eq. (7) can be simplified by eliminating the terms with zero and negative coefficients as shown in Eq. (8). Using this simplified equation, a large amount of processing time can be saved without missing any possible solutions. Hence, in our example,

$$X_0(2) + X_1(3) + X_2(1) = 0 . \tag{8}$$

Step 3. Try out all possible combinations of X_i from zero up to a pre-defined upper limit ($\pm N$). The upper limit can be defined as the sum of the coefficients in Eq. (8). Record all combinations of X_i that make Eq. (8) equal to zero. The values of X_i are the corresponding powers of I_{pq} that eliminate the rotational dependent terms in Eq. (6). For our example: $I_{30} \cdot I_{12}^3$, $I_{21}^2 \cdot I_{02}$, $I_{30}^2 \cdot I_{02}^3$, etc.

Note that the negative power of I_{pq} is represented by I_{qp}, since I_{qp} is the complex conjugate of I_{pq}.

It is very time consuming to solve Eq. (8). Especially when we are working with higher-order moment invariants, Eq. (8) gets more complicated. The process can be simplified by making use of the following theorem.

Theorem. *If a and b are positive integers that are mutually prime, there must be two integers x′, y′ that satisfy*

$$ax' + by' = 1 . \tag{9}$$

Let $x = cx'$, $y = cy'$. Then from Eq. (9), we have

$$ax + by = c(ax' + by') = c . \tag{10}$$

Then, if a and b are mutually prime and c is an integer, there must exist integer solutions that satisfy Eq. (10). One possible solution of Eq. (10) is

$$x=cx' \quad \text{and} \quad y=cy' .$$

In general, if $x=cx'$ and $y=cy'$ form one set of the solution of Eq. (10), the general solution of Eq. (10) can be written as

$$x=cx'-bn, \quad y=cy'+an \quad (n=0, \pm 1, \ldots) . \tag{11}$$

Eq. (9) can be solved by the Recursive Division and Backward Substitution (RDBS) Algorithm as shown by the following pseudo code.

Recursive Division

```
i=1
dividend_i = max{a, b}
divisor_i = min{a, b}
do{
  quotient_i = -dividend_i/divisor_i
  dividend_{i+1} = divisor_i
  divisor_{i+1} = dividend_i mod divisor_i
  i=i+1}
while (divisor_i ≠ 1)
quotient_i = 1
```

Backward Substitution

```
while (i>1){
  i=i-1
  quotient_{i-1} = quotient_{i-1} * quotient_i + quotient_{i+1}}
```

The solution of x and y are stored at the first two locations of the quotient list, i.e. quotient$_1$ and quotient$_2$.

In practice, Eq. (8) can be written as

$$\sum_i X_i C_i = 0 . \tag{12}$$

Choose any two coefficients (C_j, C_k) that are mutually prime from Eq. (12) and rewrite Eq. (12) as follows:

$$X_j C_j + X_k C_k + \sum_{i \neq j,k} X_i C_i = 0 . \tag{13}$$

Solve for X_j' and X_k' from Eq. (14) using the RDBS algorithm:

$$X_j' C_j + X_k' C_k = 1 . \tag{14}$$

Then from Eq. (11), the general solution of Eq. (13) can be written as

$$X_j = -CX_j' - n \cdot C_k, \quad X_k = -CX_k' + n \cdot C_j \tag{15}$$

where $n=0, \pm 1, \pm 2, \pm 3, \ldots$ and

$$C = \sum_{i \neq j,k} X_i C_i \quad \text{for } X_i = 0, \pm 1, \pm 2, \ldots, \pm N .$$

Step 4. Eliminate all redundancy from the results obtained from Step 3. Example: consider the moment invariant $I_{21} \cdot I_{02}^2$. Then $I_{21}^2 \cdot I_{02}^4, I_{21}^3 \cdot I_{02}^6, \ldots$ are redundant and duplicating $I_{21} \cdot I_{02}^2$.

Intuitively, some of the moment invariants such as $I_{11}, I_{20} \cdot I_{02}, I_{21} \cdot I_{12}, I_{30} \cdot I_{03}$ cannot be generated by Eq. (8). However, these simple types of invariant can easily be found separately and we do not need to generate them by solving Eq. (8). By making use of this approach, the following set of equations has been generated.

For moments up to third order:

$$I_{11}, I_{20} \cdot I_{02}, I_{21} \cdot I_{12}, I_{30} \cdot I_{03}, I_{21}^2 \cdot I_{02}, I_{30} \cdot I_{12} \cdot I_{02} , \tag{16a}$$

$$I_{30} \cdot I_{12}^3, I_{30} \cdot I_{21} \cdot I_{02}^2, I_{30}^2 \cdot I_{02}^3, I_{30} \cdot I_{12}^5 \cdot I_{02} . \tag{16b}$$

For moments up to fourth order:

$$I_{22}, I_{31} \cdot I_{13}, I_{40} \cdot I_{04}, I_{31} \cdot I_{02}, I_{40} \cdot I_{13}^2, I_{40} \cdot I_{02}^2, I_{31} \cdot I_{12}^2 , \tag{16c}$$

$$I_{40} \cdot I_{13} \cdot I_{02}, I_{40} \cdot I_{03} \cdot I_{12}, I_{31} \cdot I_{03} \cdot I_{21}, I_{40} \cdot I_{31} \cdot I_{03}^2 , \tag{16d}$$

$$I_{40} \cdot I_{13} \cdot I_{12}^2, I_{40} \cdot I_{13}^3 \cdot I_{20}, I_{40} \cdot I_{03}^2 \cdot I_{20}, I_{04} \cdot I_{12}^2 \cdot I_{02} , \tag{16e}$$

$$I_{31}^2 \cdot I_{03} \cdot I_{12}, I_{40} \cdot I_{13} \cdot I_{03} \cdot I_{21}, I_{40} \cdot I_{03} \cdot I_{21} \cdot I_{02} , \tag{16f}$$

$$I_{31} \cdot I_{03} \cdot I_{12} \cdot I_{20}, \ldots . \tag{16g}$$

Step 5. The final step is to express I_{pq} in terms of the central moments μ_{pq}. Recall Eq. (36) and Eq. (38) defined in (Hu, 1962). From

$$\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

we have

$$u=U+V, \qquad v=-\mathrm{i}(U-V)\,, \tag{17}$$

$$(I_{p0},\ldots,I_{0p})(U,V)^p \equiv (\mu_{p0},\ldots,\mu_{0p})(u,v)^p\,. \tag{18}$$

Substituting Eq. (17) and the required moment order p into Eq. (18), the set of equations as shown in Eq. (20) can be obtained. Let us take the second order as an example. Substituting $p=2$ and Eq. (17) into Eq. (18), we have

$$\begin{aligned}(I_{20},I_{11},I_{02})(U,V)^2 &\equiv (\mu_{20},\mu_{11},\mu_{02})(u,v)^2\\ &\equiv \mu_{20}(U+V)^2-\mathrm{i}2\mu_{11}(U+V)(U-V)\\ &\quad -\mu_{02}(U-V)^2\\ &\equiv (\mu_{20}-\mathrm{i}2\mu_{11}-\mu_{02})U^2+(\mu_{20}+\mu_{02})2UV\\ &\quad +(\mu_{20}+\mathrm{i}2\mu_{11}-\mu_{02})V^2\,.\end{aligned} \tag{19}$$

Comparing the coefficients of Eq. (19), it is clear that

$$I_{20}=(\mu_{20}-\mu_{02})-\mathrm{i}2\mu_{11}\,,$$

$$I_{11}=(\mu_{20}+\mu_{02})\,,$$

$$I_{02}=(\mu_{20}-\mu_{02})+\mathrm{i}2\mu_{11}\,.$$

Then the relationship between the I_{xx} listed in Eq. (16) and the corresponding central moments can be obtained as follows,

$$I_{20}=(\mu_{20}-\mu_{02})-\mathrm{i}2\mu_{11}, \tag{20a}$$

$$I_{11}=\mu_{20}+\mu_{02}, \tag{20b}$$

$$I_{02}=(\mu_{20}+\mu_{02})+\mathrm{i}2\mu_{11}, \tag{20c}$$

$$I_{30}=(\mu_{30}-3\mu_{12})-\mathrm{i}(3\mu_{21}-\mu_{03})\,, \tag{20d}$$

$$I_{21}=(\mu_{30}+\mu_{12})-\mathrm{i}(\mu_{21}+\mu_{03})\,, \tag{20e}$$

$$I_{12}=(\mu_{30}+\mu_{12})+\mathrm{i}(\mu_{21}+\mu_{03})\,, \tag{20f}$$

$$I_{03}=(\mu_{30}-3\mu_{12})+\mathrm{i}(3\mu_{21}-\mu_{03})\,, \tag{20g}$$

$$I_{40}=(\mu_{40}-6\mu_{22}+\mu_{04})-\mathrm{i}4(\mu_{31}-\mu_{13})\,, \tag{20h}$$

$$I_{31}=(\mu_{40}-\mu_{04})-\mathrm{i}2(\mu_{31}+\mu_{13})\,, \tag{20i}$$

$$I_{22}=(\mu_{40}+2\mu_{22}+\mu_{04})\,, \tag{20j}$$

$$I_{13}=(\mu_{40}-\mu_{04})+\mathrm{i}2(\mu_{31}+\mu_{13})\,, \tag{20k}$$

$$I_{04}=(\mu_{40}-6\mu_{22}+\mu_{04})+\mathrm{i}4(\mu_{31}-\mu_{13})\,. \tag{20l}$$

The absolute moment invariants, which are invariant to rotation of images, can be obtained by substituting Eq. (20) into Eq. (16) and taking the real part of the results. Similarly, the skew invariants, which are useful in distinguishing mirror images, can be obtained by taking the imaginary part of the results described above. In addition, Eq. (16) can be normalized to make them invariant under a scale change by substituting the normalized central moments η_{pq}, Eq. (3), for μ_{pq}. Part of the equations of moment invariants are tabulated in Table 1.

3. Weighting factors

One method for increasing the performance of similarity measures involves a statistical analysis of the training set of features (Hais, 1981; Cash and Hatamian, 1987). Those features which are found to be more reliable than others are given more importance when making classifications. For the Euclidean distance measure, weighting factors are determined which cause the more reliable feature to make a larger contribution to the distance between two feature vectors. The weighted Euclidean distance between two feature vectors is given by

$$D_j(\Psi)=\left(\sum_{i=1}^{n} w_i(x_{ji}-\psi_i)^2\right)^{1/2} \tag{21}$$

where w_i is the ith weighting factor for the ith feature vector component. The following weighting factors were defined in (Cash and Hatamian, 1987):

$$w_i=\sigma_i, \quad w_i=\frac{1}{\bar{\sigma}_i}, \quad w_i=\frac{\sigma_i}{|\mu_i|}, \quad w_i=\frac{\sigma_i}{\bar{\sigma}_i} \tag{22}$$

where σ_i is the standard deviation of the ith feature of all classes in the training set, $\bar{\sigma}_i$ is the mean of the standard deviation of the ith feature of each class (A–Z, a–z, 0–9), and $|\mu_i|$ is the magnitude of the mean of the ith feature of all classes.

In addition to the weighting factors defined in (Cash and Hatamian, 187), there are also many possible versions of weighting factors that can be used. The following are some examples that have not been described in (Hais, 1981; Cash and Hatamian, 1987):

$$w_i=\bar{\sigma}_i, \quad w_i=\frac{1}{\sigma_i}, \quad w_i=\frac{1}{|\mu_i|}, \quad w_i=\frac{\bar{\sigma}_i}{|\mu_i|} \tag{23a}$$

$$w_i=\frac{1}{\bar{\mu}_i}, \quad w_i=\frac{\sigma_i}{\bar{\mu}_i}, \quad w_i=\frac{\bar{\sigma}_i}{\bar{\mu}_i} \tag{23b}$$

Table 1
Equations of moment invariants

Second-order moment invariants		
$\Phi_{2,1}=I_{11}$	$\Phi_{2,2}=I_{20}\cdot I_{02}$	
Third-order moment invariants		
$\Phi_{3,1}=I_{21}\cdot I_{12}$	$\Phi_{3,2}=I_{30}\cdot I_{03}$	$\Phi_{3,3}=\mathrm{Re}\{I_{21}^2\cdot I_{02}\}$
* $\Phi_{3,4}=\mathrm{Im}\{I_{21}^2\cdot I_{02}\}$	* $\Phi_{3,5}=\mathrm{Re}\{I_{30}\cdot I_{12}\cdot I_{02}\}$	* $\Phi_{3,6}=\mathrm{Im}\{I_{30}\cdot I_{12}\cdot I_{02}\}$
$\Phi_{3,7}=\mathrm{Re}\{I_{30}\cdot I_{12}^3\}$	$\Phi_{3,8}=\mathrm{Im}\{I_{30}\cdot I_{12}^3\}$	* $\Phi_{3,9}=\mathrm{Re}\{I_{30}\cdot I_{21}\cdot I_{02}^2\}$
* $\Phi_{3,10}=\mathrm{Im}\{I_{30}\cdot I_{21}\cdot I_{02}^2\}$	* $\Phi_{3,11}=\mathrm{Re}\{I_{30}^2\cdot I_{02}^3\}$	* $\Phi_{3,12}=\mathrm{Im}\{I_{30}^2\cdot I_{02}^3\}$
* $\Phi_{3,13}=\mathrm{Re}\{I_{30}\cdot I_{12}^5\cdot I_{02}\}$	* $\Phi_{3,14}=\mathrm{Im}\{I_{30}\cdot I_{12}^5\cdot I_{02}\}$	
Samples of fourth-order moment invariants		
$\Phi_{4,1}=I_{22}$	$\Phi_{4,2}=I_{31}\cdot I_{13}$	$\Phi_{4,3}=I_{40}\cdot I_{04}$
$\Phi_{4,4}=\mathrm{Re}\{I_{31}\cdot I_{02}\}$	$\Phi_{4,5}=\mathrm{Im}\{I_{31}\cdot I_{02}\}$	* $\Phi_{4,6}=\mathrm{Re}\{I_{40}\cdot I_{13}^2\}$
* $\Phi_{4,7}=\mathrm{Im}\{I_{40}\cdot I_{13}^2\}$	$\Phi_{4,8}=\mathrm{Re}\{I_{40}\cdot I_{02}^2\}$	$\Phi_{4,9}=\mathrm{Im}\{I_{40}\cdot I_{02}^2\}$
* $\Phi_{4,10}=\mathrm{Re}\{I_{31}\cdot I_{12}^2\}$	* $\Phi_{4,11}=\mathrm{Im}\{I_{31}\cdot I_{12}^2\}$	* $\Phi_{4,12}=\mathrm{Re}\{I_{40}\cdot I_{13}\cdot I_{02}\}$
* $\Phi_{4,13}=\mathrm{Im}\{I_{40}\cdot I_{13}\cdot I_{02}\}$	* $\Phi_{4,14}=\mathrm{Re}\{I_{40}\cdot I_{03}\cdot I_{12}\}$	* $\Phi_{4,15}=\mathrm{Im}\{I_{40}\cdot I_{03}\cdot I_{12}\}$
* $\Phi_{4,16}=\mathrm{Re}\{I_{31}\cdot I_{03}\cdot I_{21}\}$	* $\Phi_{4,17}=\mathrm{Im}\{I_{31}\cdot I_{03}\cdot I_{21}\}$	* $\Phi_{4,18}=\mathrm{Re}\{I_{40}\cdot I_{31}\cdot I_{03}^2\}$
* $\Phi_{4,19}=\mathrm{Im}\{I_{40}\cdot I_{31}\cdot I_{03}^2\}$	* $\Phi_{4,20}=\mathrm{Re}\{I_{40}\cdot I_{13}\cdot I_{12}^2\}$	* $\Phi_{4,21}=\mathrm{Im}\{I_{40}\cdot I_{13}\cdot I_{12}^2\}$
* $\Phi_{4,22}=\mathrm{Re}\{I_{40}\cdot I_{13}^3\cdot I_{20}\}$	* $\Phi_{4,23}=\mathrm{Im}\{I_{40}\cdot I_{13}^3\cdot I_{20}\}$	* $\Phi_{4,24}=\mathrm{Re}\{I_{40}\cdot I_{03}^2\cdot I_{20}\}$
* $\Phi_{4,25}=\mathrm{Im}\{I_{40}\cdot I_{03}^2\cdot I_{20}\}$	* $\Phi_{4,26}=\mathrm{Re}\{I_{40}\cdot I_{12}^2\cdot I_{02}\}$	* $\Phi_{4,27}=\mathrm{Im}\{I_{40}\cdot I_{12}^2\cdot I_{02}\}$
* $\Phi_{4,28}=\mathrm{Re}\{I_{31}^2\cdot I_{03}\cdot I_{12}\}$	* $\Phi_{4,29}=\mathrm{Im}\{I_{31}^2\cdot I_{03}\cdot I_{12}\}$	* $\Phi_{4,30}=\mathrm{Re}\{I_{40}\cdot I_{13}\cdot I_{03}\cdot I_{21}\}$
* $\Phi_{4,31}=\mathrm{Im}\{I_{40}\cdot I_{13}\cdot I_{03}\cdot I_{21}\}$	* $\Phi_{4,32}=\mathrm{Re}\{I_{40}\cdot I_{03}\cdot I_{21}\cdot I_{02}\}$	* $\Phi_{4,33}=\mathrm{Im}\{I_{40}\cdot I_{03}\cdot I_{21}\cdot I_{02}\}$
* $\Phi_{4,34}=\mathrm{Re}\{I_{31}\cdot I_{03}\cdot I_{12}\cdot I_{20}\}$	* $\Phi_{4,35}=\mathrm{Im}\{I_{31}\cdot I_{03}\cdot I_{12}\cdot I_{20}\}$	...
Samples of fifth-order moment invariants		
$\Phi_{5,1}=I_{32}\cdot I_{23}$	$\Phi_{5,2}=I_{41}\cdot I_{14}$	$\Phi_{5,3}=I_{50}\cdot I_{05}$
$\Phi_{5,4}=\mathrm{Re}\{I_{41}\cdot I_{03}\}$	$\Phi_{5,5}=\mathrm{Im}\{I_{41}\cdot I_{03}\}$	$\Phi_{5,6}=\mathrm{Re}\{I_{32}\cdot I_{12}\}$
$\Phi_{5,7}=\mathrm{Im}\{I_{32}\cdot I_{12}\}$	$\Phi_{5,8}=\mathrm{Re}\{I_{32}^2\cdot I_{13}\}$	$\Phi_{5,9}=\mathrm{Im}\{I_{32}^2\cdot I_{13}\}$
$\Phi_{5,10}=\mathrm{Re}\{I_{32}^2\cdot I_{02}\}$	$\Phi_{5,11}=\mathrm{Im}\{I_{32}^2\cdot I_{02}\}$	* $\Phi_{5,12}=\mathrm{Re}\{I_{50}\cdot I_{14}\cdot I_{13}\}$
* $\Phi_{5,13}=\mathrm{Im}\{I_{50}\cdot I_{14}\cdot I_{13}\}$	* $\Phi_{5,14}=\mathrm{Re}\{I_{50}\cdot I_{14}\cdot I_{02}\}$	* $\Phi_{5,15}=\mathrm{Im}\{I_{50}\cdot I_{14}\cdot I_{02}\}$
* $\Phi_{5,16}=\mathrm{Re}\{I_{50}\cdot I_{23}\cdot I_{04}\}$	* $\Phi_{5,17}=\mathrm{Im}\{I_{50}\cdot I_{23}\cdot I_{04}\}$	* $\Phi_{5,18}=\mathrm{Re}\{I_{50}\cdot I_{04}\cdot I_{12}\}$
* $\Phi_{5,19}=\mathrm{Im}\{I_{50}\cdot I_{04}\cdot I_{12}\}$	* $\Phi_{5,20}=\mathrm{Re}\{I_{50}\cdot I_{13}\cdot I_{03}\}$	* $\Phi_{5,21}=\mathrm{Im}\{I_{50}\cdot I_{13}\cdot I_{03}\}$
* $\Phi_{5,22}=\mathrm{Re}\{I_{50}\cdot I_{03}\cdot I_{02}\}$	* $\Phi_{5,23}=\mathrm{Im}\{I_{50}\cdot I_{03}\cdot I_{02}\}$	* $\Phi_{5,24}=\mathrm{Re}\{I_{41}\cdot I_{32}\cdot I_{04}\}$
* $\Phi_{5,25}=\mathrm{Im}\{I_{41}\cdot I_{32}\cdot I_{04}\}$	...	

* Moment invariants that have never been reported in the literature.

where $\overline{\mu_i}$ is the mean of the absolute values of the ith feature of all classes in the training set.

The method used for finding the weighting factors described in this section has some disadvantages. It cannot screen out those features that have very little discrimination power, and the weighting factors obtained may not be the best set. It is because the range of values that have been tried out were quite narrow. Actually, there are infinitely many combinations that can be used as weighting factors. Our next objective is to find out the most optimum set of weighting factors that can achieve the highest possible recognition rate, and to eliminate those feature vectors that have no discriminating power. We introduce a solution to the problem in the next section.

4. Automatic searching of weighting factor

One of our objectives is to develop an automatic searching approach to find the most optimum set of weighting factors. The approach tries out a large range of numbers that can be used as weighting factors for the classification process. The initial arrangement is to have points (i.e. possible weighting factors) with large separations, and each point is tested individually. Every feature vector is tested using different values of weighting factors, and all points that produce the respective highest recognition rates (coarse local maximums) were recorded. After all of the feature vectors have been tested, a few local maximum points may be located.

After the coarse local maximums are found, a fine trimming procedure is carried out in order to find the precise locations of the local maximums. It continues to iterate until it finds the best weighting factor that produces the highest recognition rate with respect to each vector. This is a Breadth First Search algorithm that can prevent it to concentrate on a local maximum and to miss the global maximum. In addition, if it is found that the recognition rate does not change by varying the weighting factor of a certain feature vector, the smallest value will be selected as the best weighting factor. For such arrangement, the weighting factors for certain feature vectors converge and tend to zero after a few iterations. Hence, those feature vectors that have very little discriminating power can be screened out.

5. Experiment

In order to test the performance of the moment invariants developed in Section 2, fifty test documents were created for this purpose. Each document consisted of sixty-two alphanumeric characters A–Z, a–z, and 0–9 as shown in Figs. 1 and 2. The characters on the test documents were aligned on separate square lattices so that the requirement for isolation of words was not critical in the experiment. Twenty of them, with right aligned characters as shown in Fig. 1, were used as training sets. The remaining test documents on which the characters were randomly rotated as shown in Fig. 2, were used as test sets in order to test the rotational invariant property of the moment invariants.

A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	a
b	c	d	e	f	g	h	i	j
k	l	m	n	o	p	q	r	s
t	u	v	w	x	y	z	0	1
2	3	4	5	6	7	8	9	

Fig. 1. Training document.

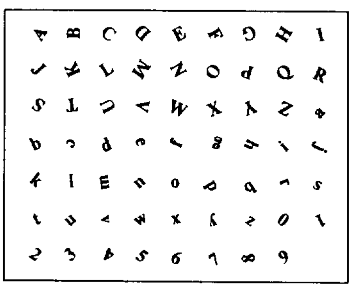

Fig. 2. Testing document.

In the experiment, most of the moment invariant expressions developed in the previous section were tested and used as features for classification. In order to eliminate the influence of scale change of the image, the normalized central moments η_{pq} were used in Eq. (20). The effect of the possible weighting factors, described in Section 3, on the performance of the classification process was also tested. The eleven weighting factors in Eqs. (22) and (23) were evaluated from the twenty training sets. Then, the performances of the weighed Euclidean distances were evaluated by using Eq. (21). Three sets of moment invariants were used in the experiment, the first set consisted of Hu's invariants, the second set of Hu's and fourth-order invariants, and the third set of our new third-order invariants (see Table 1). The recognition rates using the three sets of moment invariants and using the weighted Euclidean distances are shown in Tables 2, 3, and 4, respectively.

After the weighting factors described in Section 3 had been obtained, the process of the automatic searching of weighting factors described in Section 4 was carried out. The best weighting factors found for different sets of moment invariants are listed in Table 5. The recognition rates obtained using different numbers of training sets for the three sets of moment invariants are tabulated in Table 6.

A similar approach has also been used to find the best weighting factors for both central moments and normalized central moments. One hundred percent recognition rate can be obtained when the central moments and normalized moments (for images without rotation etc.) were used as feature vectors. For moment invariants, the recognition rates have also been improved, but not to the extent as great as the other two kinds of moments by the best set of weighting factors. It is due to the inherent property of moment invariants, that they are invariant to the rotation of images. Note carefully that the moment

Table 2
Recognition rate obtained by Hu's invariants

Weighting factor	No. of training set & Recognition rate (%)							
	1	2	3	4	5	10	15	20
None	68.47	74.12	79.43	81.55	82.63	85.51	87.52	88.72
$\bar{\sigma}$	61.36	68.56	73.88	76.97	78.41	83.98	85.23	86.68
$1/\bar{\sigma}$	74.43	79.18	82.47	85.09	85.51	88.87	90.23	91.41
σ	63.78	71.38	76.01	77.97	79.72	83.59	86.41	87.78
$1/\sigma$	71.89	77.54	81.98	83.56	85.38	86.78	89.04	90.57
$\sigma/\bar{\sigma}$	68.41	74.16	80.12	83.07	84.23	87.12	88.65	90.07
$1/\|\mu\|$	75.77	81.53	84.46	87.05	87.49	89.88	91.23	92.56
$\sigma/\|\mu\|$	73.92	79.01	82.86	83.78	86.75	89.07	89.53	91.15
$\bar{\sigma}/\|\mu\|$	74.74	78.34	82.07	83.22	85.89	88.29	89.18	90.23
$1/\bar{\mu}$	75.69	82.87	86.23	87.69	88.34	90.45	91.48	92.90
$\sigma/\bar{\mu}$	74.10	79.99	82.81	84.36	87.01	89.24	89.89	91.33
$\bar{\sigma}/\bar{\mu}$	72.02	78.68	82.19	85.37	85.79	88.80	89.61	91.19

Table 3
Recognition rate obtained by Hu's and fourth-order moment invariants

Weighting factor	No. of training set & Recognition rate (%)							
	1	2	3	4	5	10	15	20
None	68.41	76.44	83.12	85.26	87.51	92.11	93.43	94.20
$\bar{\sigma}$	62.14	70.06	77.34	81.03	83.14	87.96	90.26	91.17
$1/\bar{\sigma}$	85.87	90.68	93.29	95.32	95.55	97.29	98.03	98.29
σ	61.96	73.15	79.08	81.64	83.80	89.11	90.01	92.18
$1/\sigma$	83.12	88.94	93.07	94.44	95.41	97.03	97.85	97.83
$\sigma/\bar{\sigma}$	72.09	78.51	85.43	86.49	88.58	93.01	94.46	95.34
$1/\|\mu\|$	65.23	73.50	80.23	82.23	83.82	89.01	90.98	92.42
$\sigma/\|\mu\|$	84.27	88.19	91.13	93.21	93.52	96.28	97.14	97.90
$\bar{\sigma}/\|\mu\|$	79.88	85.52	90.43	92.18	93.22	96.67	97.49	98.34
$1/\bar{\mu}$	88.40	92.56	95.11	96.26	96.46	97.42	98.18	98.36
$\sigma/\bar{\mu}$	77.89	81.93	86.84	89.31	90.37	94.46	95.60	97.01
$\bar{\sigma}/\bar{\mu}$	73.79	79.82	85.28	87.19	88.75	94.04	94.82	96.44

invariants will not be able to distinguish the character pairs, such as (6, 9), (b, q) and (d, p).

6. Conclusion

From the experimental results listed in Tables 2–4, it is evident that the performance of all sets of moment invariants has been greatly improved by the use of suitable weighting factors. The most useful weighting factors for Hu's and the new third-order moment invariants were $1/|\mu|$ and $1/\bar{\mu}$. When the fourth-order moment invariants were used, over 98% recognition rate could be obtained by using the weighting factors $1/\bar{\sigma}$, $\bar{\sigma}/|\mu|$ and $1/\bar{\mu}$. When more training sets were used, the recognition rate was found to be higher than those obtained by both sets of the third-order moment invariants.

After automatic searching of the best weighting factors, Hu's third-order moment invariants together with the fourth-order moment invariants were tested using the best weighting factors listed in Table 5. The recognition rate of using this combination was found to be significantly better than that of using Hu's third-order moment invariants alone. But after the new third-order moment invariants were used, an even

Table 4
Recognition rate obtained by new third-order moment invariants

Weighting factor	No. of training set & Recognition rate (%)							
	1	2	3	4	5	10	15	20
None	67.96	74.23	80.11	81.76	83.26	86.34	87.90	89.58
$\bar{\sigma}$	57.15	63.78	69.21	71.91	73.66	78.70	81.31	83.42
$1/\bar{\sigma}$	82.85	87.32	90.20	92.94	93.16	95.15	96.12	96.70
σ	61.58	66.87	72.43	75.23	76.56	81.37	83.49	85.52
$1/\sigma$	80.87	84.74	88.83	91.06	91.62	93.88	95.10	95.81
$\sigma/\bar{\sigma}$	72.81	77.46	82.02	83.77	85.42	87.91	89.39	91.20
$1/\vert\mu\vert$	88.17	91.50	95.13	95.54	96.24	97.48	97.79	97.91
$\sigma/\vert\mu\vert$	78.08	82.10	84.47	86.49	88.47	91.34	91.81	93.29
$\bar{\sigma}/\vert\mu\vert$	77.09	81.73	84.21	86.73	88.50	90.91	91.78	93.14
$1/\bar{\mu}$	88.17	91.52	95.11	95.67	96.35	97.48	97.78	97.91
$\sigma/\bar{\mu}$	78.07	82.12	84.64	86.51	88.55	91.22	91.83	93.28
$\bar{\sigma}/\bar{\mu}$	76.92	81.68	84.21	86.71	88.56	90.90	91.76	93.13

Table 5
Best weighting factors for moment invariants

Hu's invariants	Weighting factor	Hu's and 4th-order	Weighting factor	New 3rd-order	Weighting factor
Φ_1	0.2	Φ_2	1.0	$\Phi_{2,1}$	0.008
Φ_2	1.0	Φ_3	10.0	$\Phi_{2,2}$	0.032
Φ_3	12.5	Φ_4	31.25	$\Phi_{3,1}$	1.56
Φ_4	0.0	Φ_6	125.0	$\Phi_{3,2}$	0.64
Φ_5	0.0	Φ_7	10000.0	$\Phi_{3,4}$	4.0
Φ_6	2500.0	$\Phi_{4,2}$	1.0	$\Phi_{3,6}$	10.0
Φ_7	15600.0	$\Phi_{4,3}$	1.0	$\Phi_{3,9}$	12.5
		$\Phi_{4,9}$	100.0	$\Phi_{3,10}$	10.0
		others	0.0	$\Phi_{3,11}$	100.0
				$\Phi_{3,12}$	125.0
				others	0.0

Noted that $\Phi_1, ..., \Phi_7$ in Table 5 are moment invariants defined in (Hu, 1962).

Table 6
Recognition rate of moment invariants using the best weighting factors

Feature used	No. of training set & Recognition rate (%)							
	1	2	3	4	5	10	15	20
Hu's invariants	86.06	89.58	93.01	94.12	94.32	95.90	96.12	96.33
Hu's & 4th-order	91.71	93.26	95.63	96.40	96.82	97.78	98.26	98.54
New 3rd-order	91.75	94.55	96.42	96.78	96.92	97.41	98.46	98.80

higher recognition rate could be obtained by using them as feature vectors.

For real time applications, the use of the new third-order moment invariants is preferable, since their recognition rate is not only better than that of the fourth-order moments but their computation time is also shorter. Using the fast computation algorithm described in (Siu et al., 1992), the average computation time required for the computation of the fourth-order moments of all the 62 characters in a 512×512 image as shown in Figs. 1 and 2 is 83.17 ms. But the computation time required for the corresponding third-order moments (including the new third-order moments) is only 49.72 ms.

The improvement of recognition rate of the present study is due to (i) our approach for the generation of new moment invariants, and (ii) our suggested approach for the selection of best weighting factors. The speed of realization of moments has also been greatly improved by our recently suggested approach (Siu et al., 1992) for using ROM look-up tables for fast realization of raw moments. We found that both the new third-order and fourth-order moment invariants can significantly improve the recognition rate for character recognition. However, further usefulness of these two new sets of moment invariants or the selection of useful moment invariants from these sets and the assignment of weighting factors for other applications, such as industrial objects, islands or aircrafts, are a fruitful area for further investigation.

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