Best Paper Award

presented to
Xu Chen
Francis C. M. Lau
Department of Electronic and Information Engineering
Hong Kong Polytechnic University, Hong Kong

for the paper entitled:

Construction of High-Rate $QL$-LDPC Codes

Dr. Trần Xuân Nam
GENERAL CHAIR

Dr. Matthias Pätzold
TPC CHAIR
To: Prof. Francis C. M. Lau

Department of Electronic and Information Engineering
Hong Kong Polytechnic University, Hong Kong

August 3, 2011

Dear Prof. Lau,

Our warmest congratulations on your following excellent work presented at the 2011 International Conference on Advanced Technologies for Communications (ATC/REV 2011):

“Construction of High-Rate QC-LDPC Codes”

by

Xu Chen and Francis C. M. Lau

As usual, the review of the ATC/REV 2011 underwent a careful process in order to select top quality research papers. Immediately upon the nominations for the best paper award made by two anonymous reviewers, we have additionally consulted an external expert in the field. He acknowledged the high quality of the work in form of a letter of recommendation. Based on all the collected information, the Technical Program Committee made then the final decision to present the Best Paper Award of ATC/REV 2011 to your colleague and you for the paper mentioned above.

Apart from the actual award’s certificate that was presented to you during the award ceremony as part of the Conference Gala Dinner, we also would like to send you the formal letter of recommendation, which reads as follows:

“Low-Density Parity-Check (LDPC) codes are of central importance to future communication standards due to their good performance and flexible design that makes it possible to adapt them to every relevant communication scenario. For practical implementability, structured LDPC codes with an explicit construction are to be preferred over random designs. It is therefore of essential interest to further the theory and design for structured LDPC codes so as
to bring their performance to par with random codes, as well as guarantee a minimum error floor at higher SNR.

The present paper considers the design of quasi-cyclic (QC) LDPC codes related to array codes, a structured LDPC code design that has been widely studied. The design of such codes for high rate, however, is a known open problem and requires methods for code shortening that maximize the girth of the associated decoding graph. The paper builds on past work by Fossorier (2004) and by Milenkovic, Kashyap and Leyba (2006) and provides a number of important new results, including lower and upper bounds on the number of columns retained in the design of girth $8$, $10$, and above codes, and a general lower bound applying to all girths.

In doing so, the paper contributes fundamentally to the advancement of communication technology, providing communication engineers with the tools to design codes for high rate applications (e.g. optical communications).

It is therefore my pleasure to wholeheartedly recommend the paper for the Best Paper Award at ATC 2011.”

Dr. Jossy Sayir
Cambridge University, UK.

Once again, we thank you sincerely for your excellent scientific contribution, not only to the conference per se, but also to the advancement in the field of communications in general.

Best regards,

Tran Xuan Nam
General Chair

Nguyen Linh-Trung
TPC Co-Chair

Matthias Pätzold
TPC Co-Chair
Construction of High-Rate QC-LDPC Codes

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Abstract—In this paper, we construct high-rate quasi-cyclic low-density parity-check (QC-LDPC) codes with girth-10 based on shortened array codes. First, we derive analytic results on the maximum number of columns for shortened array codes of different girths. A code construction method inspired by the analysis is proposed for column-weight-three codes and is compared with the conventional greedy construction algorithm. We show that the proposed method is more effective than the conventional greedy algorithm in the sense that the minimum length of the codes constructed using our proposed method to achieve different code rates is comparative or much shorter than those constructed using the greedy construction.

I. INTRODUCTION

Quasi-cyclic low-density parity-check (QC-LDPC) codes form a class of structured LDPC codes with the parity-check matrices consisting of circulant permutation sub-matrices [1] [2] [3]. The application of QC-LDPC codes to optical communications has been extensively investigated recently, see [4] and references therein. One of the challenges in the next generation optical communication systems is to find channel codes with low redundancy along with extremely low error floor [5]. Constructing large-girth QC-LDPC codes is one promising solution to achieve the objective [6]–[8]. However, how to systematically construct high-rate QC-LDPC codes with a girth of ten or higher is still an open problem [9].

There have been some works on constructing QC-LDPC codes of large girth. The shortened array codes has been first investigated in [8], where only a subset of columns in array codes are retained to avoid the “cycle-governing” equations corresponding to various cycle lengths. For shortened array codes, the maximum number of columns that remain after shortening plays an essential role in the range of the applicable code rate. The larger the number of remaining columns is, the higher the code rate is achieved.

In this paper, we derive a general lower bound of the maximum size of retaining columns for shortened array codes and upper bounds of that for codes with girth-eight, girth-ten or above. The theoretical analysis provides us much insight, leading to a construction method for column-weight-three shortened array codes. We further compare our method with the conventional greedy algorithm [8].

Section II reviews the basics of shortened array codes and the previously established results. Section III presents our analysis on the shortened array codes. Section IV focuses on the special case of column-weight-three shortened array codes. A code construction method and some results are presented in the same section. Section V concludes the paper.

II. REVIEW OF ARRAY CODES

The general form for the parity-check matrix of an array code [10] is represented by

$$ H = \begin{bmatrix} I & P^{a_0} & \cdots & P^{a_0(p-1)} \\ I & P^{a_1} & \cdots & P^{a_1(p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ I & P^{a_{r-1}} & \cdots & P^{a_{r-1}(p-1)} \end{bmatrix}, $$

(1)

where $p$ denotes the number of columns and is a prime number; $r$ is the number of block rows with $1 \leq r \leq p$; $a_g$’s are distinct numbers with $0 \leq a_g \leq p-1$, for $g = 0, \ldots, r-1$; $I$ is the identity matrix and $P$ is a $p \times p$ circulant permutation matrix defined as

$$ P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. $$

(2)

Since each column of $H$ has a weight of $r$ and each row has a weight of $p$, the code rate $R$ is lower bounded by $R \geq 1 - r/p$. Throughout the paper, we call $a_g$ the block-row indices, where $0 \leq g \leq r-1$; and $h$ the block-column indices, where $0 \leq h \leq p-1$.

According to [1, Theorem 2.1], a cycle of length $2k$ associated with a sequence of circulant permutation matrices $P^{a_{g_1}h_1}, P^{a_{g_1}h_2}, P^{a_{g_2}h_2}, \ldots, P^{a_{g_k}h_k}, P^{a_{g_k}h_1}$ exists if and only if

$$(a_{g_1} - a_{g_2})h_1 + (a_{g_2} - a_{g_3})h_2 + \cdots + (a_{g_k} - a_{g_1})h_k \equiv 0 \pmod{p},$$

(3)

where the block-row indices $a_{g_2}$ to $a_{g_1}$, and the block-column indices $h_1$ to $h_k$ of the permutation matrices satisfy $a_{g_l} \neq a_{g_{l+1}}$ and $h_l \neq h_{l+1}$, for $l = 1, 2, \ldots, k-1$, $a_{g_k} \neq a_{g_1}$, and $h_k \neq h_1$. Thus, in order to avoid cycles of length eight, the selection of the block-row indices $\{a_0, a_1, \ldots, a_{r-1}\}$ in a shortened array code must take into account the following property [8].

**Inevitability of cycle-eight due to block-row indices:** For $a_{g_1}, a_{g_2}, a_{g_3}, a_{g_4} \in \{a_0, a_1, \ldots, a_{r-1}\}$, cycle-eight must exist if the following equality holds,

$$ a_{g_1} + a_{g_3} - a_{g_2} - a_{g_4} = 0 \pmod{p} $$

subject to $a_{g_1} \neq a_{g_2}, a_{g_1} \neq a_{g_4}, a_{g_2} \neq a_{g_3}, a_{g_3} \neq a_{g_4}$. (4)

Once the set of block-row indices $\{a_0, a_1, \ldots, a_{r-1}\}$ has been selected, the block-column indices can be regarded as variables and the cycle-governing equations in the form of (3)
can be written as
\[
\sum_{i=1}^{k} c_i x_i \equiv 0 \pmod{p},
\]
where the coefficient \( c_i = a_{g_i} - a_{g_2} \) if \( i = 1 \) and \( c_i = a_{g_i} - a_{g_{i+1}} \) if \( i \neq 1 \). Hence, the equation \( \sum_{i=1}^{k} c_i = 0 \) is always satisfied. An example showing the systems of cycle-governing equations corresponding to different sets of block-row indices is given in Table I. Any solution \( \mathbf{x} = (x_1, x_2, \cdots, x_k) \) to (5) with \( x_i \in \{0, 1, \cdots, p-1\} \) and \( x_i \neq x_j \) for all \( i \neq j \) is referred to as a proper solution over \( \mathbb{Z}_p \), where \( \mathbb{Z}_p \) is the ring of integers modulo \( p \). In the construction of shortened array codes, we aim to find a subset of block-column indices \( \mathcal{S}(p; \Omega) \subseteq \{0, 1, \cdots, p-1\} \) such that \( \mathcal{S}(p; \Omega) \) contains no proper solutions to a system of cycle-governing equations denoted by \( \Omega \).

**Definition 1:** An equation \( \sum_{i=1}^{k} c_i x_i \equiv 0 \pmod{p} \) over \( \mathbb{Z}_p \)
is of type \((l, m)\) if \( l \) coefficients are positive and \( m \) are negative with \( l + m = k \). Note that type \((l, m)\) is equivalent to type \((m, l)\).

For example, all the cycle-six governing equations involve three variables, and they are of type \((1, 2)\), i.e.,
\[
\Omega : (c_{m_1} + c_{m_2}) x_3 \equiv c_{m_1} x_1 + c_{m_2} x_2 \pmod{p}, \quad m_1, \ldots, |\Omega|.
\]

We further denote by \( s(p; \Omega) \) the maximum number of columns retained in a shortened array code. Intuitively, \( s(p; \Omega) \) is closely related to the range of the code rate that shortened array codes can achieve. In [8], the authors have characterized \( s(p; \Omega) \) for a certain set of cycle-governing equations. In particular, they have derived a lower bound of \( s(p; \Omega) \) for shortened array codes of girth-eight by slightly modifying the Behrend’s construction method [11].

**Theorem 1:** (Lower Bound of \( s(p; \Omega) \) for shortened array codes of girth-eight [8]) Let \( \Omega \) denote the system of cycle-six governing equations in the form of (6) and let \( V = \max_{m \in \{1, \ldots, |\Omega|\}} \{c_{m_2} + c_{m_1}\} \). Then \( s(p; \Omega) \) is lower-bounded by
\[
s(p, \Omega) \geq \gamma_1 P e^{-\gamma_2 \sqrt{\log p - \frac{1}{2} \log \log P} + (1 + o(1))}
\]
where \( \gamma_1 = V^2 \sqrt{\frac{1}{2}} \log V, \quad \gamma_2 = 2V^2 \log V \) and \( o(1) \) is a term vanishing as \( p \to \infty \).

However, how \( s(p; \Omega) \) scales for shortened array codes of different girths has not been derived. In the following, we will first derive a general lower bound of \( s(p; \Omega) \) for shortened array codes of various girths. Then we derive upper bounds of \( s(p; \Omega) \) for codes with girth-eight, and girth-ten and above, respectively.

### III. ANALYSIS OF THE SHORTENED ARRAY CODES

#### A. General Lower Bound of \( s(p; \Omega) \)

**Theorem 2** (General Lower Bound): Given a system of cycle-governing equations \( \Omega \) up to cycle-2k, there exists a sequence \( s_1, s_2, \cdots, s_n \) with \( 1 = s_1 < s_2 < \cdots < s_n \leq |\Omega|k(n-1)k^{-1} \) such that \( \mathcal{S}(p; \Omega) = \{s_1, s_2, \cdots, s_n\} \) does not contain proper solutions to \( \Omega \). Then \( s(p; \Omega) \) for shortened array codes of girth-2 \((k+1)\) is lower-bounded by
\[
s(p; \Omega) \geq \frac{p^k}{k!} \left( |\Omega| \right) - \frac{1}{k!} + 1.
\]

**Proof:** The proof is based on the idea of the greedy construction of \( \mathcal{S}(p; \Omega) \) specified in [12, Theorem 2.1] and we sketch the proof here.

Obviously we can choose \( s_1 = 1 \). Assume that a sequence \( s_1, s_2, \cdots, s_{n-1} \) has been chosen such that the sequence does not contain any proper solutions to \( \Omega \). Now we want to find a \( s_n \) such that adding it to the sequence would not create any proper solutions to \( \Omega \). Since the cycle-2k equations involve at most \( k \) variables, we want to find \( s_n \) satisfying
\[
c_{m,i_0} x_{i_0} \neq - \sum_{1 \leq i \leq k, i \neq i_0} c_{m, i_0} x_i \pmod{p},
\]
for all \( 1 \leq i \leq k \) and \( i \neq i_0 \). We choose \( i_0 \) such that \( x_{i_0} \) is distinct integers and \( x_i \in \{s_1, \ldots, s_{n-1}\} \). For a fixed \( i_0 \) and \( m, x_i (1 \leq i \leq k, i \neq i_0) \) can be taken from at most \((k-1)!(n-1)^{-1}k^{-1}\) possible values and hence the constraint function (7) excludes no more than \((k-1)!\) values for \( s_n \). Considering that there are \( k \) possible values of \( i_0 \) and \( |\Omega| \) possible values of \( m \), the cycle-governing equations are at most \( k|\Omega|^{-1}(k-1)! \) possible values for \( s_n \). Since \( k|\Omega|^{-1}(k-1)! < k|\Omega|^{-1}(n-1)^{-1}k^{-1} \), we can always find a \( s_n < k|\Omega|^{-1}(n-1)^{-1}k^{-1} \) so that adding it to the set would not create proper solutions.

Following this process, we can extend the set \( \mathcal{S}(p; \Omega) = \{s_1, s_2, \cdots, s_n\} \) to the extent that \( \frac{p^k}{k!} \left( |\Omega| \right) - \frac{1}{k!} + 1 \). Hence, \( s(p; \Omega) \geq n \geq \frac{p^k}{k!} \left( |\Omega| \right) - \frac{1}{k!} + 1 \).
B. $s(p; \Omega)$ for Shortened Array Codes of Girth-eight

A lower bound of $s(p; \Omega)$ for shortened array codes of girth-eight has been derived in [8]. Here we will derive an upper bound of $s(p; \Omega)$.

Theorem 3: (Upper Bound of $s(p; \Omega)$ for shortened array codes of girth-eight) Let $\Omega$ denote the system of cycle-six governing equations in the form of (6). Then $s(p; \Omega)$ is upper-bounded by $s(p; \Omega) \leq p(\log \log p)^{-c(v)}$, where $c(v) = 2^{-2^{v+10}} + v = \min_{m \in \{1, \ldots, |\Omega|\}} \{c_{m,2} + c_{m,1}\}$.

Proof: Let $v_m = c_{m,1} + c_{m,2}$ and $S(p; \Omega)$ be the set of column indices avoiding the proper solutions to $\Omega$. Then $S(p; \Omega)$ must not contain an arithmetic progression of length $v_m + 1$. We prove this by contradiction, showing that it would contain proper solutions to (6).

Assume $S(p; \Omega)$ contains an arithmetic progression of length $v_m + 1$. Then we can take three variables from $S(p; \Omega)$ such that $x_3 = x_1 + c_{m,2}d$ and $x_2 = x_1 + (c_{m,1} + c_{m,2})d$, where $d$ is the common difference of successive members in the arithmetic progression. Obviously $x_1, x_2$ and $x_3$ are distinct integers that satisfy the equation $(c_{m,1} + c_{m,2})x_3 \equiv c_{m,1}x_1 + c_{m,2}x_2$. It contradicts the fact that $S(p; \Omega)$ avoids the proper solutions.

By making use of [13, Theorem 1.3] that any subset of $\{1, \ldots, n\}$ of size not less than $p(\log \log p)^{-c(v_m)}$ contains an arithmetic progression of length $v_m + 1$, where $c(v_m) = 2^{-2^{v_m+10}}$, we can prove that $s(p; \Omega) \leq p(\log \log p)^{-c(v_m)}$. Since the inequality holds for all $m = 1, 2, \ldots, |\Omega|$, taking $v = \min_{m \in \{1, \ldots, |\Omega|\}} \{c_{m,2} + c_{m,1}\}$ yields the tightest upper bound for $s(p; \Omega)$ as stated in the theorem.

The upper bound of $s(p; \Omega)$ derived above is of almost the same order as the lower bound of $s(p; \Omega)$ in Theorem 1 except for the sublinear terms. In particular, by combining both theorems we can obtain the asymptotic order of $s(p; \Omega)$ for girth-eight shortened array codes as $\lim_{p \to \infty} \frac{\log s(p; \Omega)}{\log p} = 1$.

C. $s(p; \Omega)$ for Shortened Array Codes with a Girth of Ten and Above

In this section, we consider $s(p; \Omega)$ for shortened array codes of girth-ten or higher.

Definition 2 (Symmetric Equations over $Z_p$): An equation is symmetric over $Z_p$ if the number of variables $k$ is even and the equation can be arranged as the following form

$$c_1x_1+\cdots+c_k/2x_{k/2} \equiv c_1x_{k/2+1}+\cdots+c_k/2x_k \pmod{p}. \quad (8)$$

In the following, we obtain an upper bound of the set of proper solutions to symmetric equations over $Z_p$ by slightly modifying [12, Theorem 3.2].

Lemma 1: Denote by $A$ the set in which there is no proper solution to a symmetric equation over $Z_p$ in $k$ variables, then we have $|A| \leq \sqrt{k(k-1)/2}p$ and hence $|A| = O(\sqrt{p})$.

Proof: Consider a set $A \subseteq \{0, 1, \ldots, p-1\}$ and the variables $x_i \in A$, for $i = 1, \ldots, k$. Let $l(n)$ be the number of solutions of $c_1x_1+\cdots+c_k/2x_{k/2} = n \pmod{p}$. Then $\sum_{n=0}^{p-1} l(n) = |A|^{k/2}$ and the total number of solutions to (8) is $\sum_{n=0}^{p-1} t^2(n)$.

Consider the solutions to (8) in which $x_i = x_j$ for certain $1 \leq i < j \leq k$, the number of such solutions is not more than $|A|^{k-2}$. Taking into account the $\binom{k}{2}$ possible choices $i$ and $j$, the total number of solutions containing at least two identical values is not more than $\binom{k}{2}|A|^{k-2}$. For a set $A$ containing only non-proper solutions (i.e., no proper solutions) to (8) over $Z_p$, it must satisfy $\sum_{n=0}^{p-1} t^2(n) \leq \binom{k}{2}|A|^{k-2}$. On the other hand, we can apply the Cauchy-Schwarz inequality to the lower bound $\sum_{n=0}^{p-1} t^2(n)$ and obtain $p \sum_{n=0}^{p-1} t^2(n) \geq (\sum_{n=0}^{p-1} l(n))^2 = |A|^k$. Therefore, we can obtain an upper bound of $|A|$ as $|A| \leq \sqrt{\frac{k(k-1)}{2}} p$.

Since the girth-eight governing equations must include at least one symmetric equation involving four variables in the form of $x_1 + x_2 \equiv x_3 + x_4 \pmod{p}$, applying $k = 4$ to Lemma 1 yields an upper bound of $s(p; \Omega)$.

Corollary 1: For shortened array codes of girth-ten or higher, $s(p; \Omega)$ is upper-bounded by $s(p; \Omega) \leq \sqrt{kp}$ and $s(p; \Omega) = O(\sqrt{p})$.

IV. CONSTRUCT COLUMN-WEIGHT-THREE SHORTENED ARRAY CODES WITH GIRTH-TEN

For the case of girth-ten codes, Theorem 2 gives a lower bound scaling as $\Theta(p^2)$ while Corollary 1 gives an upper bound scaling as $\Theta(p^{3/2})$. Therefore, either one bound or both bounds are loose. In the following, we will show that when the shortened array codes have a column weight of three, the lower bound given in Theorem 2 is loose and we will derive a much tighter lower bound for this case. Although this is merely a special case of array codes, it is useful in searching for large-girth high-rate QC-LDPC codes [8], [9].

Theorem 4: Given a column-weight-three array code, $s(p; \Omega)$ for shortened array codes of girth-ten is lower bounded by $s(p; \Omega) \geq \sqrt{kp} - \alpha \sqrt{k\log p}$ for some positive $\beta$ and $\alpha$.

Proof: Omitted due to shortage of space.

According to Theorem 4, $\lim_{p \to \infty} \frac{s(p; \Omega)}{p^{3/2}} = \infty$, for any $\epsilon > 0$. It can be observed that this lower bound is almost at the same order of the upper bound given in Corollary 1 and thus is much tighter than that given in Theorem 2.

Constructing high-rate QC-LDPC codes of girth-ten or higher is a challenging problem. In this section, we propose a code construction algorithm and compare it with the greedy algorithm in [8]. Another code construction method, namely the random code construction, is also considered to evaluate the effectiveness of different code construction algorithms.

Algorithm - Proposed Code Construction Method

1. Let $V = \max_{m=1,2,\cdots,|\Omega|} \{c_{m,1} + c_{m,2}\}$. For the largest prime number $q$ with $q < \sqrt{kp}$, do the following steps:

2. Construct a set $X = \{x' + V x : 0 \leq x' \leq q-1, x' = x^2 \pmod{p}\}$.

3. Initialize $S(p; \Omega)$ as the largest subset of $X$ that avoids proper solutions to cycle-governing equations in four variables. ($X$ is guaranteed to avoid proper
solutions to all cycle-governing equations in three variables.)

3) Update $S(p; \Omega)$ by sequentially adding the integers in $\{0, p - 1\} \setminus S(p; \Omega)$ that would not create proper solutions to $\Omega$.

The comparison of the minimum $p$ required to achieve the same code rate $R$ using the greedy code construction method [8] and our proposed method is shown in Table II. It can be seen that our proposed code construction can achieve the same code rate using either a comparable $p$ or a much smaller $p$ compared with the greedy construction method.

In order to further evaluate the effectiveness of the code construction algorithm, we compare it with a random code construction, which is based on heuristic computer search. Given a submatrix size $p$ and a maximum iteration number $M$, the random code construction updates $S(p; \Omega)$ by randomly adding integers in $\{0, \ldots, p - 1\} \setminus S(p; \Omega)$ that would not create proper solutions to $\Omega$. It stops when all the integers in $\{0, \ldots, p - 1\}$ have been exhausted. The same process repeats for $M$ iterations and it outputs the maximum achievable code rate. In Table II, we show the achievable code rate using random construction with $M = 500$ iterations. Moreover, $p$ is chosen as those found by the greedy construction method and our proposed method. The results have indicated that (i) increasing the value of $p$ may not produce a higher code rate (first case); (ii) the achievable code rate may be further enhanced by the random construction method (second case); and (iii) the achievable code rate is not enhanced in most cases when the smaller one between $p_{\text{greedy}}$ and $p_{\text{proposed}}$ is used in the random construction method.

V. CONCLUSION

We have derived a general lower bound of the maximum size of retaining columns for shortened array codes and upper bounds for codes with girth-eight, girth-ten or above, which is closely related to the range of code rate that shortened array codes can achieve. We have proposed a method of constructing high-rate QC-LDPC of girth-ten with column-weight-three. We have shown that the proposed method is more effective than the conventional greedy algorithm in the sense that the minimum length of the codes constructed using our proposed method to achieve different code rates is comparative or much shorter than those constructed using the greedy construction.

ACKNOWLEDGEMENTS

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REFERENCES


TABLE II: Comparison of the minimum $p$ required to construct girth-ten shortened array codes for different code rates $R$ using the greedy code construction [8] ($p_{\text{greedy}}$) and our proposed method ($p_{\text{proposed}}$). The random construction method involves 500 iterations. The column weight of the codes is $r = 3$ and the block-row indices is $\{0, 1, 3\}$.

<table>
<thead>
<tr>
<th>Code Rate</th>
<th>Minimum $p$ required</th>
<th>Random Code Construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = 0.727$</td>
<td>$p_{\text{proposed}} = 911$</td>
<td>$p = p_{\text{proposed}} : R = 0.727$</td>
</tr>
<tr>
<td></td>
<td>$p_{\text{greedy}} = 1039$</td>
<td>$p = p_{\text{greedy}} : R = 0.727$</td>
</tr>
<tr>
<td>$R = 0.75$</td>
<td>$p_{\text{proposed}} = 1319$</td>
<td>$p = p_{\text{proposed}} : R = 0.737$</td>
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<tr>
<td></td>
<td>$p_{\text{greedy}} = 1439$</td>
<td>$p = p_{\text{greedy}} : R = 0.777$</td>
</tr>
<tr>
<td>$R = 0.77$</td>
<td>$p_{\text{proposed}} = 1723$</td>
<td>$p = p_{\text{proposed}} : R = 0.786$</td>
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<tr>
<td></td>
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<td>$p = p_{\text{greedy}} : R = 0.777$</td>
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<td>$R = 0.842$</td>
<td>$p_{\text{proposed}} = 4493$</td>
<td>$p = p_{\text{proposed}} : R = 0.842$</td>
</tr>
<tr>
<td></td>
<td>$p_{\text{greedy}} = 4861$</td>
<td>$p = p_{\text{greedy}} : R = 0.85$</td>
</tr>
</tbody>
</table>