

Reference

Vector Analysis

Chapter 2

Static Electric Fields (3 Weeks)

Chapter 3.3 Coulomb's Law

Chapter 3.4 Gauss's Law and Applications

Chapter 3.5 Electric Potential

Chapter 3.6 Material Media in Static Electric Field

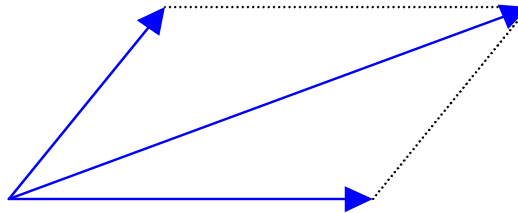
Chapter 3.8 Boundary conditions for Electrostatic Fields

Chapter 3.9 Capacitance and Capacitors

III. VECTOR ANALYSIS

3.1 Vector algebra

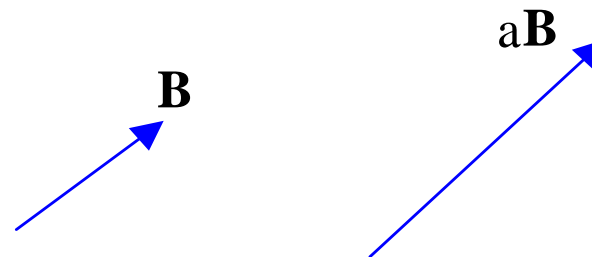
- addition



- associative law $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

- commutative law $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

- multiplication by scalar



- distributive law $a(\mathbf{B} + \mathbf{C}) = a\mathbf{B} + a\mathbf{C}$

- Scalar (or Dot) Product:

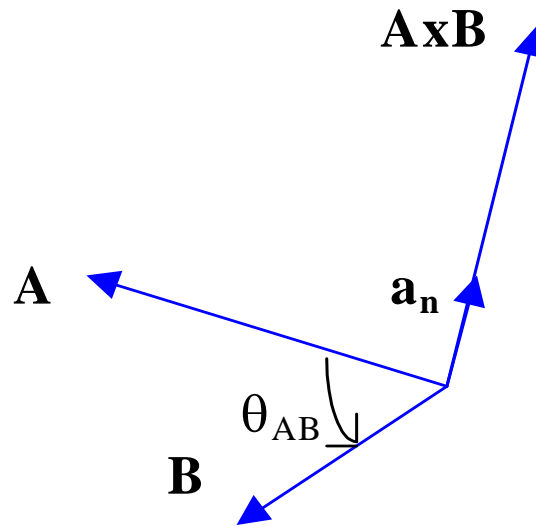
$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta_{AB}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

- Vector (or Cross) Product:

$$\mathbf{A} \times \mathbf{B} \equiv \mathbf{a}_n |\mathbf{A}| |\mathbf{B}| \sin(\theta_{AB})$$



Note for cross products,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \neq \mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

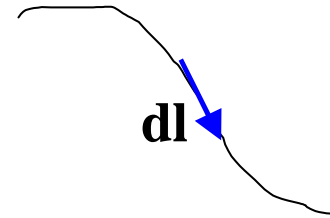
Cartesian Co-ordinate System (unit vectors \mathbf{a}_x \mathbf{a}_y \mathbf{a}_z)

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Differential displacement vector $d\mathbf{l}$

$$d\mathbf{l} = dx \mathbf{a}_x + dy \mathbf{a}_y + dz \mathbf{a}_z$$



Example: To integrate \mathbf{F} along the path $y=x^2$ from a (1,1) to b (2,4), where $\mathbf{F} = y\mathbf{a}_x + \mathbf{a}_y$

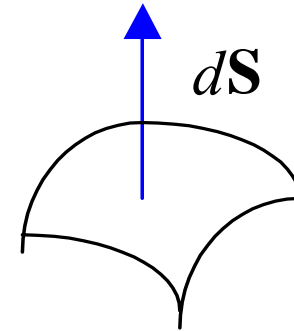
$$y = x^2$$

$$\int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_a^b (y\mathbf{a}_x + \mathbf{a}_y)(dx\mathbf{a}_x + dy\mathbf{a}_y)$$

$$= \int_1^2 y dx + \int_1^4 dy = 5 \frac{1}{3}$$

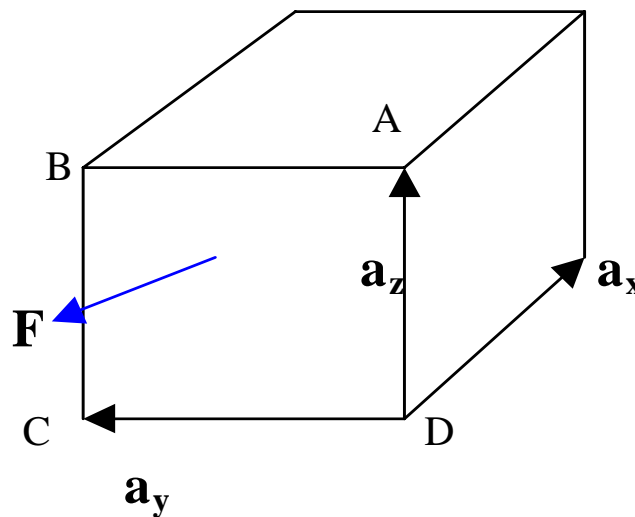
Differential surface vector $d\mathbf{S}$:

+ve in the 'outward' normal direction to the surface element for an enclosed volume



Example:

Find $\mathbf{F} \cdot d\mathbf{S}$ for the surface ABCD for the given \mathbf{F} in the following unit cube.



$$\mathbf{F} = (2y + z)\mathbf{a}_x - 2\mathbf{a}_y$$

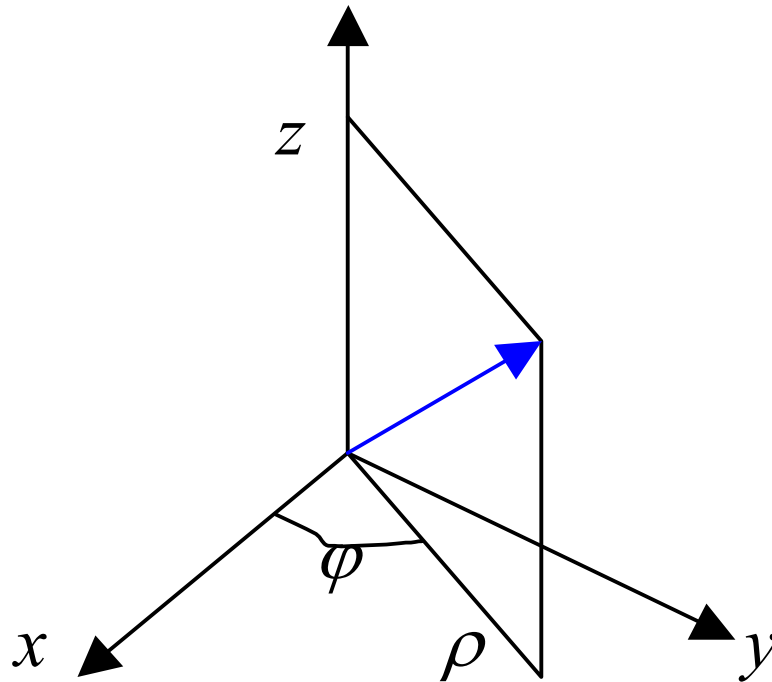
$$d\mathbf{S} = -dydz \mathbf{a}_x \quad \text{for surface ABCD}$$

$$\int \mathbf{F} \cdot d\mathbf{S} = - \int_0^1 \int_0^1 (2y + z) dy dz$$

$$= - \int_0^1 \left[2yz + \frac{z^2}{2} \right]_0^1 dy = -\frac{3}{2}$$

Note the minus sign in $d\mathbf{S}$ because positive direction is outward from the enclosed volume

Cylindrical coordinate system



$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \left\{ \frac{y}{x} \right\}$$

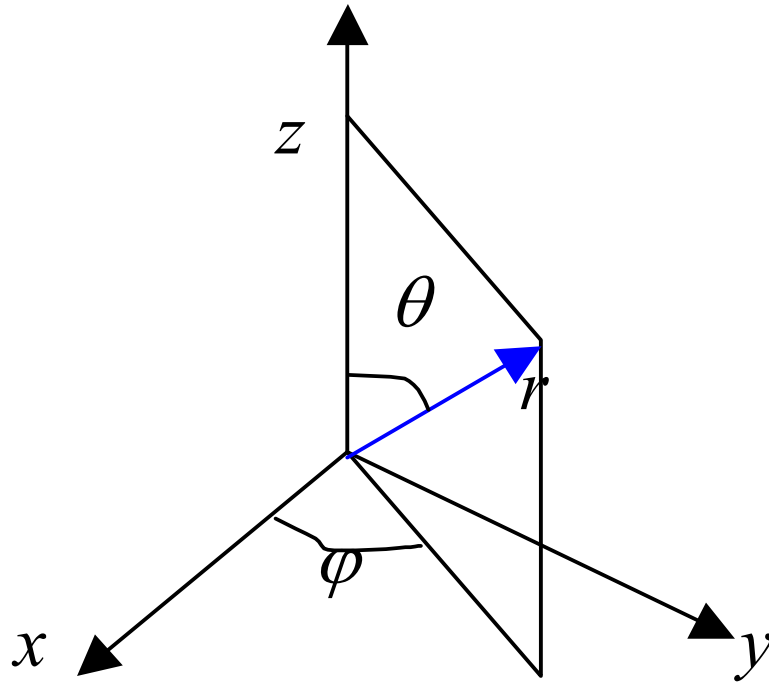
$$z = z$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$$

Spherical coordinate system



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1} \left\{ \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\}$$

$$\phi = \tan^{-1} \left\{ \frac{y}{x} \right\}$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

$$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$$

3.2 Scalar and Vector Calculus

Integrals of scalars and vectors over volumes, surfaces and lines often used in electromagnetics. Examples:

ρ_v is the charge density (per unit volume)

\mathbf{D} is the electric flux density (or electric displacement)

\mathbf{E} is the electric field intensity

$$\int_V \rho_v dv \quad \text{Total charge enclosed within volume } V$$

$$\int_S \mathbf{D} \cdot d\mathbf{S} \quad \text{Total electric flux passing through the surface } S$$

$$\int_C \mathbf{E} \cdot d\mathbf{l} \quad \text{Potential difference between two points on the line}$$

Gradient of scalar field f

$$\mathbf{grad} f = \nabla f = \frac{\partial f}{\partial x} \mathbf{a}_x + \frac{\partial f}{\partial y} \mathbf{a}_y + \frac{\partial f}{\partial z} \mathbf{a}_z$$

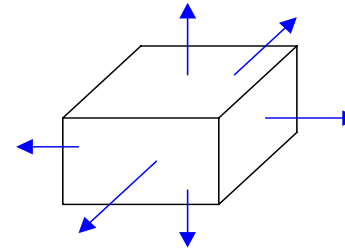
$\mathbf{grad} f$ is a vector in the direction of maximum increase of the field f .

$$\nabla f = \left. \frac{\partial f}{\partial l} \right|_{\max} \mathbf{a}_n$$

\mathbf{a}_n is unit vector in the direction of maximum increase of f

Divergence of a vector field \mathbf{A} :

$$\operatorname{div} \mathbf{A} \equiv \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$



If we consider \mathbf{A} as a flux density (per unit surface area), the closed surface integral represents the net flux leaving the volume Δv

In rectangular coordinates,

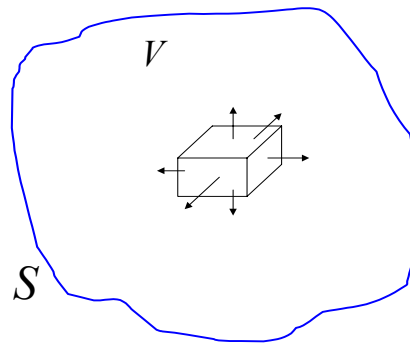
$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence Theorem:

If \mathbf{A} is a vector, then for a volume V surrounded by a closed surface S ,

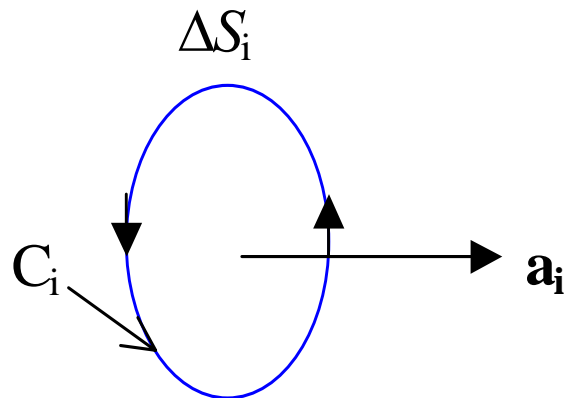
$$\int_V \nabla \cdot \mathbf{A} \, dv = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

The above integral represents the net flux leaving the closed surface S if \mathbf{A} is the flux density



Curl of a vector field: Curl of a vector \mathbf{A} is a measure of the tendency of \mathbf{A} to “push or pull” around a closed path that encircles a point. Component of $\text{curl}\mathbf{A}$ in direction \mathbf{a}_i

$$(\text{curl } \mathbf{A})_i \equiv (\nabla \times \mathbf{A})_i \equiv \lim_{\Delta S_i \rightarrow 0} \frac{\oint_{C_i} \mathbf{A} \cdot d\mathbf{l}}{\Delta S_i}$$



Since the maximum value of any component of a vector is equal to the magnitude of the vector,

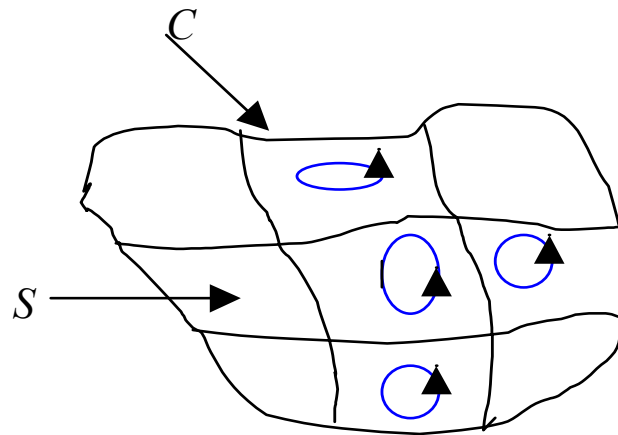
$$|\nabla \times \mathbf{A}| = \mathbf{a}_n \left[\lim_{\Delta S \rightarrow 0} \frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max}$$

In rectangular coordinates

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial & \partial & \partial \\ \partial x & \partial y & \partial z \\ A_x & A_y & A_z \end{vmatrix}$$

Stokes's theorem: For an open surface S bounded by a contour C ,

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$



The line integrals from adjacent cells cancel leaving the only the contribution along the contour C which bounds the surface S .

IV. ELECTROSTATIC FIELDS (Time-Invariant)

The electric field intensity \mathbf{E} is defined as the force on an unit positive charge at a point.

\mathbf{D} is the electric flux density (or electric displacement density). The direction of \mathbf{D} is that of the electric flux at the point, and its magnitude is the no.of flux lines passing through an unit normal surface area.

4.1 Gauss's Law :

$$\nabla \cdot \mathbf{D} = \rho_v$$

Point form

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

Integral form

ρ_v is the space volume charge density (coulomb per unit volume) at a point. Q is the total charge enclosed within a closed surface S . Unit of \mathbf{D} is C/m^2 .

Relation between \mathbf{D} and \mathbf{E} :

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}$$

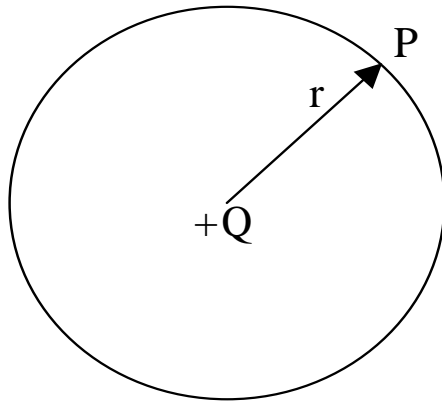
ϵ_0 is permittivity of free space (vacuum), ϵ_r is the relative permittivity (or dielectric constant) of the medium material, and ϵ is the permittivity of the medium material. (Unit of ϵ_0 and ϵ is Farad/metre)

In free space,

$$\epsilon_r = 1$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

- Gauss's law can determine the electric field pattern due to any distribution of charges.
- In the simplest case, consider the field at a point P of distance r from a point charge of magnitude Q in free space.



Construct a spherical surface with centre at Q and radius r . \mathbf{E} is the same everywhere on the surface. Applying integral form of Gauss's law gives

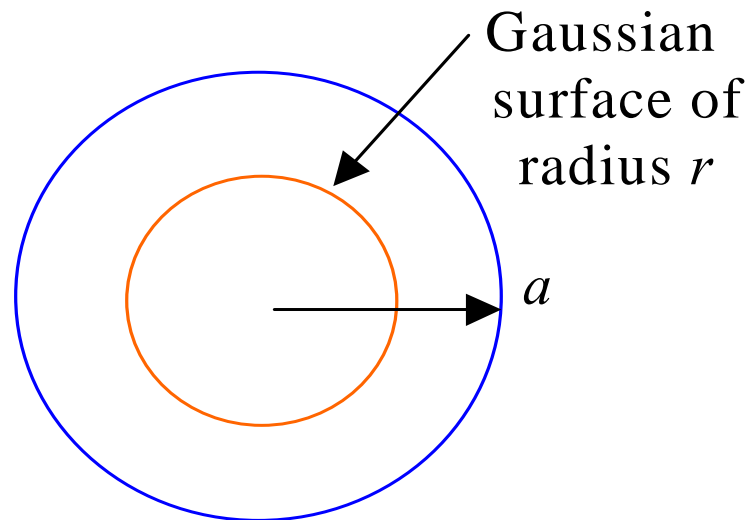
$$4\pi r^2 |\mathbf{E}| = \frac{Q}{\epsilon_0}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

The force acting a test charge q at point P is then (Coulomb's law):

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \mathbf{a}_r$$

Example : Find the electric field due to a spherical charge distribution of radius a with uniform volume charge density ρ_v in free space.



Solution:

Case (i): For a Gaussian surface of radius r less than the radius a of the charge distribution, the total charge enclosed by the Gaussian surface is:

$$Q = \frac{4}{3} \pi r^3 \rho_v$$

Applying Gauss's Law to the Gaussian surface,

$$4 \pi r^2 |\mathbf{E}(r)| = \frac{Q}{\epsilon_0}$$

$$\mathbf{E}(r) = \frac{r \rho}{3 \epsilon_0} \mathbf{a}_r$$

Case (ii): For a Gaussian surface of radius r greater than a :

$$Q = \frac{4}{3} \pi a^3 \rho$$

Applying Gauss's law,

$$4\pi r^2 |\mathbf{E}(r)| = \frac{Q}{\epsilon_0}$$

$$|\mathbf{E}(r)| = \frac{a^3 \rho}{3\epsilon_0 r^2} \mathbf{a}_r$$

4.2 Electric (electrostatic) potential V :

The electrostatic field is conservative, i.e.

$$\nabla \times \mathbf{E} = 0$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

According to vector identity, for any scalar V

$$\nabla \times (\nabla V) \equiv 0$$

- So we can define a scalar electric potential V which is easier to work with,

$$\mathbf{E} = -\nabla V$$

(Hence unit of \mathbf{E} is volt/metre)

Integrating along a path

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$$

- Note that the integral is independent of the path taken. Physically the potential difference is equal to the work done in moving an unit charge from point A to point B.
- Potential due to N point charges:

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{Q_k}{|\mathbf{r} - \mathbf{r}_k|}$$

4.3 Capacitance

Capacitance between two conductors is defined as:

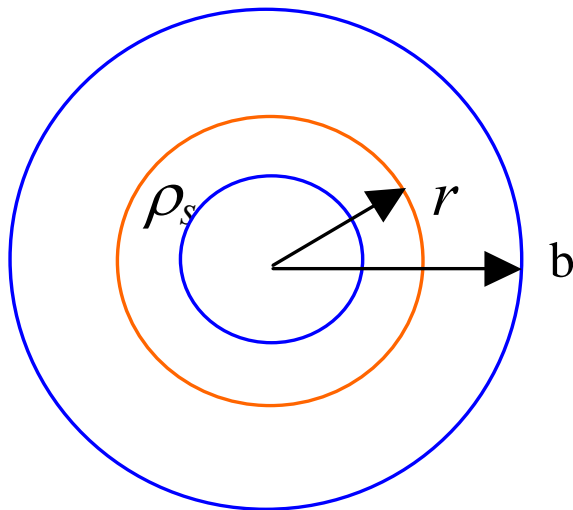
$$C = \frac{Q}{V}$$

where Q is the charge on the conductor and V is the potential difference between the conductors.

Example: Find the potential difference between two coaxial cylinders of radius a , b in the following diagram. Assume the surface charge density (per unit area) on the inner conductor is ρ_s . Hence find the capacitance per unit length of the cylinder.

Solution:

Construct a Gaussian cylindrical surface of unit length and radius r .
The \mathbf{E} -field at radius r is:



$$2\pi r |\mathbf{E}(r)| = \frac{\rho_s 2\pi a}{\epsilon}$$

$$\mathbf{E}(r) = \frac{\rho_s a}{\epsilon r} \mathbf{a}_r$$

The potential difference V_{ab} between outer and inner conductor:

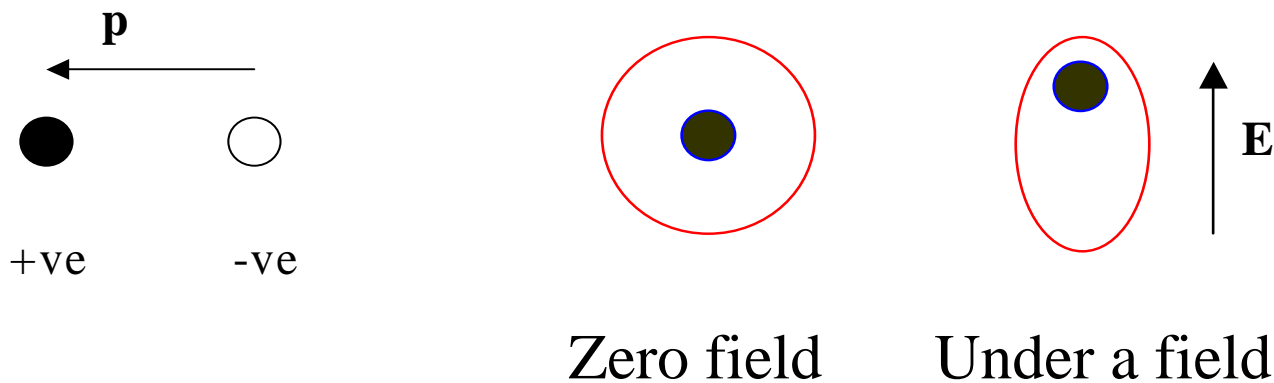
$$\begin{aligned} V_{ab} &= V_a - V_b = -\int_b^a \mathbf{E}(r) \cdot d\mathbf{r} \\ &= -\int_b^a \frac{\rho_s a dr}{\epsilon r} = \frac{\rho_s a}{\epsilon} \ln\left(\frac{b}{a}\right) \end{aligned}$$

Capacitance per unit length

$$\begin{aligned} C &= \frac{2\pi a \rho_s}{V_{ba}} \\ &= \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)} \end{aligned}$$

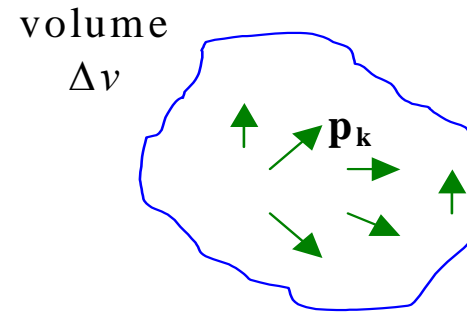
4.5 Dielectric material

- Dielectrics are insulating materials, and contain bound charges which cannot move freely to generate currents.
- Bound charges can move short distances under an electric field to form electric dipoles. (A dipole is a pair of equal and opposite charges $(+Q, -Q)$ separated at a small distance r . The dipole moment \mathbf{p} is equal to $Qr\mathbf{a}_r$.)



- From a macroscopic view point, all the minute dipole moments \mathbf{p}_k add up to result in a net dipole moment per unit volume called the polarization vector \mathbf{P} (C/m²):

$$\mathbf{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{N\Delta v} \mathbf{p}_k}{\Delta v}$$



where N is the number of dipoles per unit volume.

- The polarization (or bound) charges can also contribute to the electric field \mathbf{E} as the free charges. (Note ρ_v defined before is free charge density.) It can be shown that \mathbf{D} is related to \mathbf{E} and \mathbf{P} according to the following equation:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

- In most materials, \mathbf{P} is in the same direction and proportional to \mathbf{E} ($\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$), so that

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi \mathbf{E} \\ &= \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \end{aligned}$$

where ε_0 is the permittivity of vacuum, ε_r is the relative permittivity (or dielectric constant) of the material, ε is the permittivity of the material, and χ is the susceptibility of the material.

4.6 Presence of a conductor

Electric current will flow in a conductor when there is an electric field. The current density \mathbf{J} (A/m²) is proportional to the electric field intensity \mathbf{E} .

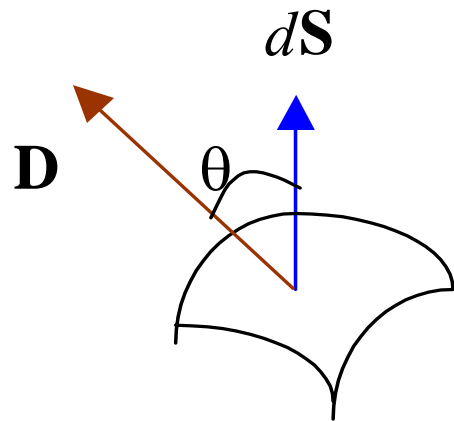
$$\mathbf{J} = \sigma\mathbf{E}$$

where σ is the conductivity of the material. This is another form of Ohm's Law. The total current I flowing through a surface S is given by:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

In the interior of a perfect conductor, σ is very large so that \mathbf{E} can be assumed to be zero.

END



$$\mathbf{D} \cdot d\mathbf{S} = |\mathbf{D}| |d\mathbf{S}| \cos \theta$$

If \mathbf{D} is the flux density (per unit surface area), $\mathbf{D} \cdot d\mathbf{S}$ represents the flux leaving the surface element $d\mathbf{S}$ in the **normal** direction. So

$\int_S \mathbf{D} \cdot d\mathbf{S}$ is the total flux leaving the entire surface S .

The physical meaning of the integral form of Gauss Law: the total electric flux leaving a closed surface

$\oint_S \mathbf{D} \cdot d\mathbf{S}$ is equal to the net charge enclosed by the

surface. The net charge is just equal to the summation of all the positive or negative charges inside. Note that the variation of the positions of the charges does NOT alter the total flux leaving the surface; the variation only affects the distribution pattern of the electric lines of force.

An **electrostatic** field is conservative, i.e. $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$

It means the energy required to move any charge around a closed loop is zero! The reason is that in moving along the path, sometimes work is done by the electric force on the charge, and sometimes work is done against the electric force.

If we regard $\int \mathbf{E} \cdot d\mathbf{l}$ as the potential

difference between two points on the path, $\oint_C \mathbf{E} \cdot d\mathbf{l}$ must be zero because it is the potential difference between the same points.

Note that $\oint_C \mathbf{E} \cdot d\mathbf{l}$ is not necessarily zero in a time-varying magnetic field because of the induced e.m.f. due to the magnetic flux.