Modulation

– Modulation is a process of encoding information from a message source in a manner suitable for transmission.

• Translating a baseband message signal to a bandpass signal at frequencies that are very high when compared to the baseband frequency.
  
  o Modulating signal or Message
  
  o Carrier wave
Introduction

– Example:
  • Mobile telephone system
    o Baseband message signal – voice signal
    o Carrier wave – 900MHz sinusoidal wave
    o Modulation method – Digital Modulation
  
  • Radio
    o Baseband message signal – audio signal
    o Carrier wave – 100MHz sinusoidal wave
    o Modulation method – Frequency Modulation
Efficient transmission

- Example: Antenna size
  - For efficient radiation, physical dimensions > 0.1 wavelength
    - Audio signal contains frequency components down to 100Hz $\Rightarrow$ wavelength = 3km ($\lambda = \frac{c}{f}$)
    - Modulated frequency at 100MHz $\Rightarrow$ wavelength = 3m
In **baseband data transmission**, a data stream represented in the form of a discrete pulse-amplitude modulated (PAM) signal is transmitted over a low-pass channel.

**Example**: Nyquist channel

\[ H(f), G(f), C(f) \]
Baseband Data Transmission

- To maximize S/N ratio at the receiver output, the matched filter is used.

- To eliminate ISI, pulse shaping techniques such as Nyquist channel is used.
  - The transmission bandwidth is

\[
W = \frac{1}{2T} = \frac{R}{2}
\]
In **passband data transmission**, the incoming data stream is modulated onto a carrier with fixed frequency and then transmitted over a band-pass channel.

**Example:**

\[ H(f), G(f), C(f) \]

\[ 2W \]

\[ \cos 2\pi f_c t \]
Passband Data Transmission

– To maximize S/N ratio at the receiver output, the matched filter is used.

– To eliminate ISI, pulse shaping techniques such as Nyquist channel is used.
  • The transmission bandwidth is

\[
2W = \frac{1}{T} = R
\]
Passband Data Transmission

Example:
- 101
- Raised cosine spectrum ($\alpha=0.5$)
- Binary Phase shift keying (BPSK)
Passband Data Transmission

Example:
- Raised cosine spectrum (\(\alpha=0.5\))
- Binary Phase shift keying (BPSK)

\[(1+\alpha)R_b = 1.5R_b\]
Types

The modulation process making the transmission possible involves switching (keying) the amplitude, frequency, or phase of a sinusoidal carrier in accordance with the incoming data.

There are three basic signaling schemes:

Amplitude-shift keying (ASK)
Frequency-shift keying (FSK)
Phase-shift keying (PSK)
Two waveforms
Mobile Telephone Systems

- GSM: Gaussian Minimum Shift Keying (GMSK) is used (a variation of FSK)
- IS-54: $\pi/4$-Differential Quaternary Phase Shift Keying (DQPSK) is used (a variation of PSK)
Unlike ASK signals, both PSK and FSK signals have a constant envelope.

PSK and FSK are preferred to ASK signals for passband data transmission over nonlinear channel (amplitude nonlinearities) such as microwave link and satellite channels.
Classification of digital modulation techniques

Coherent and Noncoherent

Digital modulation techniques are classified into coherent and noncoherent techniques, depending on whether the receiver is equipped with a phase-recovery circuit or not.

The phase-recovery circuit ensures that the local oscillator in the receiver is synchronized to the incoming carrier wave (in both frequency and phase).
Phase Recovery (Carrier Synchronization)

A local oscillator can be synchronized with an incoming carrier wave

Transmit a pilot carrier (similar to the DSB-LC modulation in analog communication)
In an $M$-ary signaling scheme, there are $M$ possible signals during each signaling interval of duration $T$.

Usually, $M = 2^n$ and $T = nT_b$ where $T_b$ is the bit duration.
In passband transmission, we have

\( M \)-ary ASK

\( M \)-ary PSK

\( M \)-ary FSK

We can also combine different methods:

\( M \)-ary quadrature-amplitude modulation (QAM)

(In baseband data transmission, we have \( M \)-ary PAM)
**M-ary signaling**

*M*-ary signaling schemes are preferred over binary signaling schemes for transmitting digital information over band-pass channels when the requirement is to conserve bandwidth at the expense of increased power.

The use of *M*-ary signaling enables a reduction in transmission bandwidth by the factor \( n = \log_2 M \) over binary signaling.

\[
T = nT_b
\]

\[10101\cdots01\] \(M\) bits \(\rightarrow\) modulator

Bandwidth \(\propto T\)

\[T\]
Coherent PSK

The functional model of passband data transmission system is

- $m_i$ is a sequence of symbol emitted from a message source.
- The channel is linear, with a bandwidth that is wide enough to transmit the modulated signal and the channel noise is Gaussian distributed with zero mean and power spectral density $N_o / 2$. 
Coherent PSK

The following parameters are considered for a signaling scheme:

**Probability of error**

A major goal of passband data transmission systems is the optimum design of the receiver so as to minimize the average probability of symbol error in the presence of additive white Gaussian noise (AWGN)

\[ P_e \]
Coherent PSK

Power spectra

Use to determine the signal **bandwidth** and **co-channel interference** in multiplexed systems.

In practice, the signalings are linear operation, therefore, it is sufficient to evaluate the **baseband** power spectral density.
Coherent PSK

Example:

- Raised cosine spectrum ($\alpha=0.5$)
- Binary Phase shift keying (BPSK)

\[(1+\alpha)R_b = 1.5R_b\]

\[2R_b\]
**Coherent PSK**

**Bandwidth Efficiency**

- Bandwidth efficiency \( \rho = \frac{R_b}{B} \) bits/s/Hz

where \( R_b \) is the data rate and \( B \) is the used channel bandwidth.

**Example:** Nyquist channel for baseband data transmission

Bandwidth \( B = W = 1/2T_b \).

\[ \therefore \rho = \frac{R_b}{B} = \frac{1}{2T_b} = 2 \text{ bits/s/Hz} \]
In a coherent binary PSK system, the pair of signals \( s_1(t) \) and \( s_2(t) \) used to represent binary symbols 1 and 0, respectively, is defined by

\[
\begin{align*}
    s_1(t) &= \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \\
    s_2(t) &= \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)
\end{align*}
\]

where \( 0 \leq t \leq T_b \), and \( E_b \) is the transmitted signal energy per bit.
Coherent PSK

Example:

\[
E = \int_0^{T_b} [s_1(t)]^2 \, dt = \frac{2E_b}{T_b} \int_0^{T_b} \cos^2 (2\pi f_c t) \, dt = \frac{2E_b}{T_b} \cdot \frac{T_b}{2} = E_b
\]

To ensure that each transmitted bit contains an integral number of cycles of the carrier wave, the carrier frequency \( f_c \) is chosen equal to \( n/T_b \) for some fixed integer \( n \).
The transmitted signal can be written as

\[ s_1(t) = \sqrt{E_b} \phi(t) \quad \text{and} \]
\[ s_2(t) = -\sqrt{E_b} \phi(t) \]

where \( \phi(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad 0 \leq t < T_b \)

Note: \( \phi^2(t) = \int_0^{T_b} \left[ \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \right]^2 dt = 1 \)
To generate a binary PSK signal, the first step is representing the input binary sequence in polar form with symbols 1 and 0 represented by constant amplitude levels of and , respectively.

This signal transmission encoder is performed by a polar nonreturn-to-zero (NRZ) encoder.

\[
S_i = \begin{cases} 
+ \sqrt{E_b} & \text{input symbol is 1} \\
- \sqrt{E_b} & \text{input symbol is 0}
\end{cases}
\]
Generation of coherent binary PSK signals

The second step is multiplying the carrier encoder output with the carrier

\[ s_i(t) = \begin{cases} 
  s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) & \text{if } s_i = \sqrt{E_b} \\
  s_2(t) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) & \text{if } s_i = -\sqrt{E_b}
\end{cases} \]

\( f_c = n / T_b \)

\[ \phi(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \]
Detection of coherent binary PSK signals

To detect the original binary sequence of 1s and 0s, we apply the noisy PSK signal to a correlator. The correlator output is compared with a threshold of zero volts.

\[
\begin{align*}
\int_0^{T_b} (\phi(t) x(t)) dt &= x_1 \\
0 &\quad \text{if } x_1 < 0 \\
1 &\quad \text{if } x_1 > 0
\end{align*}
\]
Detection of coherent binary PSK signals

Example

If the transmitted symbol is 1, \( x(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \)

and the correlator output is

\[
x_1 = \int_0^{T_b} x(t)\phi(t)dt \\
= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \cdot \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)dt \\
= \sqrt{E_b} \cdot \frac{2}{T_b} \int_0^{T_b} \cos^2 (2\pi f_c t)dt \\
= \sqrt{E_b}
\]

Similarly, if the transmitted symbol is 0, \( x_1 = -\sqrt{E_b} \)
We can represent a coherent binary system with a signal constellation consisting of two message points.

- The coordinates of the message points are all the possible correlator output under a noiseless condition.

- The coordinates for BPSK are $\sqrt{E_b}$ and $-\sqrt{E_b}$.

Decision boundary

\[ -\sqrt{E_b} \quad \sqrt{E_b} \]
Error probability of binary PSK

There are two possible kinds of erroneous decision:

- Signal $s_2(t)$ is transmitted, but the noise is such that the received signal point inside region with $x_1 > 0$ and so the receiver decides in favor of signal $s_1(t)$.

- Signal $s_1(t)$ is transmitted, but the noise is such that the received signal point inside region with $x_1 < 0$ and so the receiver decides in favor of signal $s_2(t)$. 

\[
\int_0^{T_b} \phi(t) s_i(t) + w(t) \left\{ \begin{array}{ll}
1 & \text{if } x_1 > 0 \\
0 & \text{if } x_1 < 0
\end{array} \right.
\]
Error probability of binary PSK

For the first case, the observable element \( x_1 \) is related to the received signal \( x(t) \) by

\[
x_1 = \int_0^{T_b} x(t)\phi(t)dt
\]

\[
= \int_0^{T_b} [s_i(t) + w(t)]\phi(t)dt
\]

\[
= -\sqrt{E_b} + \int_0^{T_b} w(t)\phi(t)dt
\]

\( x_1 \) is a Gaussian process with mean:

\[
\bar{x}_i = E[x_i]
\]

\[
= E[-\sqrt{E_b} + \int_0^{T_b} w(t)\phi(t)dt]
\]

\[
= -\sqrt{E_b}
\]
Error probability of binary PSK

Variance is

\[ \sigma^2 = E[(x_i - \bar{x}_i)^2] \]

\[ = E \left[ \left( \int_0^{T_b} w(t)\phi(t)dt \right)^2 \right] \]

\[ = E \left[ \int_0^{T_b} \int_0^{T_b} w(t)w(u)\phi(t)\phi(u)dtdu \right] \]

\[ = \int_0^{T_b} \int_0^{T_b} E[w(t)w(u)]\phi(t)\phi(u)dtdu \]

\[ = \int_0^{T_b} \int_0^{T_b} \frac{N_o}{2} \delta(t-u)\phi(t)\phi(u)dtdu \]

\[ = \frac{N_o}{2} \int_0^{T_b} \phi^2(t)dt \]

\[ = \frac{N_o}{2} \]

PB.34
Therefore, the conditional probability density function of $x_1$, given that symbol 0 was transmitted is

$$f(x_1 | 0) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{(x_1 - \bar{x}_1)^2}{2\sigma^2} \right]$$

$$= \frac{1}{\sqrt{\pi N_o}} \exp \left[ -\frac{(x_1 + \sqrt{E_b})^2}{N_o} \right]$$
Error probability of binary PSK

and the probability of error is

\[ p_{10} = \int_{0}^{\infty} f(x_1 | 0) dx_1 \]

\[ = \frac{1}{\sqrt{\pi N_o}} \int_{0}^{\infty} \exp\left[- \frac{(x_1 + \sqrt{E_b})^2}{N_o}\right] dx_1 \]

Putting \( z = \frac{1}{\sqrt{N_o}} (x + \sqrt{E_b}) \), we have

\[ p_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b / N_o}}^{\infty} \exp[- z^2] dz \]

\[ = \frac{1}{2} \text{erfc}\left(\frac{\sqrt{E_b}}{\sqrt{N_o}}\right) \]
Similarly, the error of the second kind

\[ p_{01} = p_{10} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_o}} \right) \]

and hence

\[ p_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_o}} \right) \]
Error probability

The probability of bit error rate is proportional to the distance between the closest points in the constellation.

BPSK

\[ P_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_o}} \right) = \frac{1}{2} \text{erfc} \left( \frac{d}{2 \sqrt{N_o}} \right) \]

Binary FSK

\[ P_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{2N_o}} \right) = \frac{1}{2} \text{erfc} \left( \frac{d}{2 \sqrt{N_o}} \right) \]
Transmission Bandwidth

The power spectral density (PSD) of the BPSK for both rectangular and raised cosine rolloff pulse shapes are plotted.

null-to-null bandwidth $= 2R_b$

$\rho = \frac{R_b}{B} = \frac{R_b}{2R_b} = 0.5 \text{ bps/Hz}$

$=(1+\alpha)R_b = 1.5R_b$
Quadriphase-shift keying (QPSK)

QPSK has twice the bandwidth efficiency of BPSK, since 2 bits are transmitted in a single modulation symbol. The data input $d_k(t)$ is divided into an in-phase stream $d_I(t)$, and a quadrature stream $d_Q(t)$.

\[
d_k(t) : 1001
\]

\[
d_I(t) : 10
\]

\[
d_Q(t) : 01
\]
QPSK

$d_k(t)$

1 0 0 1

$t$

$d_I(t)$

1 0

$t$

$d_Q(t)$

0 1

$t$

$T = 2T_b$

$T_b$
QPSK

The phase of the carrier takes on one of four equally spaced values, such as $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + (2i-1)\pi / 4] & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

where $i = 1, 2, 3, 4$.

$E$ is the transmitted signal energy per symbol;
$T$ is the symbol duration;
$f_c = n / T$;

(Note: $T = 2T_b$)
The transmitted signal can be written as

\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + (2i - 1)\pi / 4] \]

\[ = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t] \cos[(2i - 1)\pi / 4] \]

\[ - \sqrt{\frac{2E}{T}} \sin[2\pi f_c t] \sin[(2i - 1)\pi / 4] \]

\[ = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) \]

where

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos[2\pi f_c t]; \quad \phi_2(t) = \sqrt{\frac{2}{T}} \sin[2\pi f_c t] \]
QPSK

\[ s_{i1} = \sqrt{E/2} \text{ or } -\sqrt{E/2} \]
\[ s_{i2} = \sqrt{E/2} \text{ or } -\sqrt{E/2} \]
Each possible value of the phase corresponds to a unique dibit.

For example: Gray code

- only a single bit is change from one dibit to the next

```
00   01
|    |
01   11
```

QPSK
Different QPSK sets can be derived by simply rotating the constellation.
<table>
<thead>
<tr>
<th>Input binary sequence</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dibit 01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dibit 10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dibit 10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dibit 00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Odd-numbered sequence 0**

Polarity of coefficient $s_{i1}$

$b_{i1} s_{i1} \phi_1(t)$

**Even-numbered sequence**

Polarity of coefficient $s_{i2}$

$b_{i2} s_{i2} \phi_2(t)$

$s(t)$
The incoming binary data sequence is first transformed into polar form by a nonreturn-to-zero level encoder. The binary wave is next divided by means of a demultiplexer into two separate binary sequences.

The result can be regarded as a pair of binary PSK signals, which may be detected independently due to the orthogonality of $\phi_1(t)$ and $\phi_2(t)$. 
Demultiplexer

Polar NRZ

$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$

$\phi_1(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$

10101

$s_i$

$s_{1i}$

$s_{2i}$

$s(t)$
Detection of coherent QPSK signals

\[ x(t) \xrightarrow{\phi_1(t)} \int_0^T x_1 \xrightarrow{\text{Decision device}} \begin{cases} 1 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 < 0 \end{cases} \]

In-phase channel

\[ x(t) \xrightarrow{\phi_2(t)} \int_0^T x_2 \xrightarrow{\text{Decision device}} \begin{cases} 1 & \text{if } x_2 > 0 \\ 0 & \text{if } x_2 < 0 \end{cases} \]

Quadrature channel
Error probability of QPSK

The received signal is

$$x(t) = s_i(t) + w(t)$$

and the observation elements are

$$x_1 = \int_0^T x(t)\phi_1(t)dt$$

$$= \pm \sqrt{E/2} + \int_0^T w(t)\phi_1(t)dt$$

$$x_2 = \int_0^T x(t)\phi_2(t)dt$$

$$= \pm \sqrt{E/2} + \int_0^T w(t)\phi_2(t)dt$$
As a coherent QPSK is equivalent to two coherent binary PSK systems working in parallel and using two carriers that are in phase quadrature.

Hence, the average probability of \textbf{bit error in each channel} of the coherent QPSK system is

\begin{equation}
    p = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_o}} \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_o}} \right)
\end{equation}
Error probability of QPSK

As the bit error in the in-phase and quadrature channels of the coherent QPSK system are statistically independent, the average probability of a correct decision resulting from the combined action of the two channels is

\[
p_c = (1 - p)^2
\]

\[
= \left[ 1 - \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_o}} \right) \right]^2
\]

\[
= 1 - \text{erfc} \left( \sqrt{\frac{E}{2N_o}} \right) + \frac{1}{4} \text{erfc}^2 \left( \sqrt{\frac{E}{2N_o}} \right)
\]
The average probability of symbol error for coherent QPSK is therefore

\[
p_e = 1 - p_c
\]

\[= \text{erfc}\left(\sqrt{\frac{E}{2N_o}}\right) - \frac{1}{4} \text{erfc}^2\left(\sqrt{\frac{E}{2N_o}}\right)\]

\[\approx \text{erfc}\left(\sqrt{\frac{E}{2N_o}}\right) \quad \text{if} \quad E/2N_o \gg 1\]
In a QPSK system, since there are two bits per symbol, the transmitted signal energy per symbol is twice the signal energy per bit,

\[ E = 2E_b \]

and then

\[ p_e \approx \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \]
With Gray encoding, the bit error rate of QPSK is

$$\text{BER} = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{N_o}}\right)$$

Therefore, a coherent QPSK system achieves the same average probability of bit error as a coherent binary PSK system for the same bit rate and the same $E_b / N_o$ but uses only half the channel bandwidth.
Note: The probability of bit error rate is also **proportional** to the distance between the closest points in the constellation.

\[
\text{BER} = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2} \text{erfc}\left(\frac{d}{2\sqrt{N_0}}\right)
\]

\[
d = 2\sqrt{E/2} = 2\sqrt{2E_b/2} = 2\sqrt{E_b}
\]
Transmission Bandwidth

Power spectral density (PSD) of the QPSK for both rectangular and raised cosine rolloff pulse shapes:

null-to-null bandwidth = $R_b$

$$\rho = \frac{R_b}{B} = \frac{R_b}{R_b} = 1 \text{ bps/Hz}$$

$$(1+\alpha)\frac{R_b}{2} = 0.75R_b$$
During each signaling interval of duration $T$, one of the $M$ possible signals

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M} (i - 1)\right) \quad i = 1, 2, \ldots$$

is sent.

$$E = E_b \log_2 M$$

$$T = T_b \log_2 M$$
M-ary PSK

\[ s_i(t) = \sqrt{E} \cos \left[ (i-1) \frac{\pi}{2} \right] \phi_1(t) - \sqrt{E} \cos \left[ (i-1) \frac{\pi}{2} \right] \phi_2(t) \]

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t \]

\[ \phi_2(t) = \sqrt{\frac{2}{T}} \sin 2\pi f_c t \]
M-ary PSK

The signal constellation of $M$-ary PSK consists of $M$ message points which are equally spaced on a circle of radius $\sqrt{E}$. For example, the constellation of 8-ary phase-shift keying is

The average probability of symbol error is

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E}{N_o}} \sin\left(\frac{\pi}{M}\right)\right)$$

$$= \operatorname{erfc}\left(\frac{d}{2\sqrt{N_o}}\right)$$

$M \geq 4$
Transmission Bandwidth

Power spectral density (PSD) of the M-ary PSK for both rectangular and raised cosine rolloff pulse shapes:
Null-to-null bandwidth efficiency of a M-ary PSK signal:

\[
\rho = \frac{R_b}{B} = \frac{R_b}{2R_b / \log_2 M} = \frac{1}{2 \log_2 M} \text{ bps/Hz}
\]

<table>
<thead>
<tr>
<th>M</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>0.5</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
</tr>
<tr>
<td>( E_b / N_o ) (BER = 10^{-6})</td>
<td>10.5</td>
<td>10.5</td>
<td>14</td>
<td>18.5</td>
<td>23.4</td>
<td>28.5</td>
</tr>
</tbody>
</table>
Frequency shifting keying (FSK)

Similar to other passband data transmission systems, the function model of FSK is:

\[ m_i \rightarrow \text{Signal transmission encoder} \rightarrow s_i \rightarrow \text{Modulator} \rightarrow s_i(t) \rightarrow \text{Channel} \rightarrow x(t) \rightarrow \text{Detector} \rightarrow x \rightarrow \text{Signal transmission decoder} \rightarrow \hat{m} \]

Carrier signal
Binary FSK

Symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that differ in frequency.

\[
s_i(t) = \begin{cases} 
\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) & 0 \leq t \leq T_b \\
0 & \text{elsewhere}
\end{cases}
\]

where \( f_i = \frac{n+i}{T_b} \) for some fixed integer and \( i = 1,2 \)

Note: the two sinusoidal waves must be orthogonal
The transmitted signal can be written as

\[ s_1(t) = \sqrt{E_b} \phi_1(t) \text{ and} \]

\[ s_2(t) = \sqrt{E_b} \phi_2(t) \]

where \( \phi_i(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) \quad 0 \leq t < T_b \)
Therefore, the constellation of binary FSK is

\[ \sqrt{E_b}, \sqrt{2E_b}, \sqrt{E_b} \]

Decision boundary

\[ \phi_1(t), \phi_2(t) \]

Constellation of binary PSK

Decision boundary

\[ \phi(t) \]
The binary data sequence is first applied to an on-off level encoder, at the output of which symbol 1 is represented by a constant amplitude of $\sqrt{E_b}$ and symbol 0 is represented by zero volts.
Detection

The detector consists of two correlators with a common input, which are supplied with locally generated coherent signals. The correlator outputs are then subtracted.

\[
\begin{align*}
\int_0^T \phi_1(t)x(t) & \quad x_1 \\
\int_0^T \phi_2(t)x(t) & \quad x_2
\end{align*}
\]

\[
y = x_1 - x_2
\]

Decision device:

- 1 if \( y > 0 \)
- 0 if \( y < 0 \)
- 0 if \( y = 0 \)
As the decision boundary is $\phi_1(t) = \phi_2(t)$,

choose 1 if $y > 0$
choose 0 if $y < 0$
Error probability

Binary FSK

\[ P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_o}}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{d}{2\sqrt{N_o}}\right) \]

BPSK

\[ P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{E_b}}{N_o}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{d}{2\sqrt{N_o}}\right) \]
The general form of binary FSK is

\[ s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) & 0 \leq t \leq T_b \\ 0 & \text{elsewhere} \end{cases} \]

Consider \( f_1 - f_2 = 1/T_b \) and their arithmetic mean equals to \( f_c \)

\[ s(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t \pm \pi t / T_b) + \text{for symbol } 1; - \text{for symbol } 0 \]

\[ = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \cos(\pm \pi t / T_b) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_c t) \sin(\pm \pi t / T_b) \]

\[ = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \cos(\pi t / T_b) \pm \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_c t) \sin(\pi t / T_b) \]
The in-phase component is independent of the input binary wave. The power spectral density of this component consists of two delta functions. (one delta function if baseband spectrum is considered)

The quadrature component is related to the input binary wave. The power spectral density is

\[
\frac{8E_b \cos^2(\pi T_b f')}{\pi^2 (4T_b^2 f'^2 - 1)}
\]
\[ s(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \cos(\pi t / T_b) \pm \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_c t) \sin(\pi t / T_b) \]

- The average powers of the delta function adding up to one-half the total power of the binary FSK signal.

- The presence of these two discrete frequency components provides a means of synchronizing the receiver with the transmitter.
Normalized power spectral density, $S_B(f)/2E_b$

Binary PSK

Delta function (part of FSK spectrum)

Normalized frequency, $fT_b$
Introduction

- PSK is usually limited to BPSK, QPSK and 8-PSK.
- For further reducing the transmission bandwidth, QAM is used.
Example:
If digital method is used, the minimum sampling rate is 2x3kHz = 6kHz. If there are 256 levels for encoding, the data rate is 48kbps

If rectangular pulse is used, the null-to-null transmission bandwidth for BPSK is
\[ 2R_b = 96\text{kHz} \]

Transmission bandwidth for 8-PSK is
\[ \frac{2R_b}{\log_2 M} = 32\text{kHz} \]

Transmission bandwidth for 1024-QAM is
\[ \frac{2R_b}{\log_2 M} = 9.6\text{kHz} \]
The M-ary QAM signal is defined by

\[
s_k(t) = \sqrt{\frac{2E_b}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T}} b_k \sin(2\pi f_c t) \quad 0 \leq t \leq T
\]

\[
k = 0, \pm 1, \pm 2, \ldots
\]

where

\[
\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad 0 \leq t \leq T
\]

\[
\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad 0 \leq t \leq T
\]

\[
\begin{bmatrix} s_{k1} \\
 s_{k2} \end{bmatrix} = \begin{bmatrix} a_k \sqrt{E_o} \\
 b_k \sqrt{E_o} \end{bmatrix}
\]
If there are $M$ symbols and $L = \sqrt{M}$, the $M$-ary square constellation can always be viewed as the Cartesian product of a one-dimensional $L$-ary PAM constellation with itself.

Therefore, we have

$$\{a_i, b_i\} = \begin{bmatrix}
(-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\
(-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\
\vdots & \vdots & \cdots & \vdots \\
(-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1)
\end{bmatrix}$$
Example: Consider a 16-QAM, $L=4$

$$\{a_i, b_i\} = \begin{bmatrix}
(-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\
(-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\
\vdots & \vdots & \cdots & \vdots \\
(-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1)
\end{bmatrix}$$

$$= \begin{bmatrix}
(-3,3) & (-1,3) & (1,3) & (3,3) \\
(-3,1) & (-1,1) & (1,1) & (3,1) \\
(-3,-1) & (-1,-1) & (1,-1) & (3,-1) \\
(-3,-3) & (-1,-3) & (1,-3) & (3,-3)
\end{bmatrix}$$
The signal constellation is

\[ \phi_2 \]

\[ \sqrt{E_o} \quad 3\sqrt{E_o} \quad \phi_1 \]
The encoding of the message is as follows:

- Two of the four bits, namely, the left-most two bits, specify the quadrant in the constellation plane in which a message point lies. Thus, starting from the first quadrant and proceeding counterclockwise, the four quadrant are represented by 11, 10, 00, and 01.

- The remaining two bits are used to represent one of the four possible lying within each quadrant of the plane.
Error Probability

The probability of correct detection for $M$-ary QAM is
\[ P_c = (1 - P_e')^2 \]
where $P_e'$ is the probability of symbol error for the corresponding $L$-ary PAM.

The probability of symbol error for $M$-ary QAM is
\[ P_e = 1 - P_c \]
\[ = 1 - (1 - P_e')^2 \]
\[ \approx 2P_e' \]
\[ = 2\left(1 - \frac{1}{\sqrt{M}}\right)\text{erfc}\left(\sqrt{\frac{E_o}{N_o}}\right) \]
Comparison of Digital Modulation Schemes

Probability of Error

- Signaling Scheme
  - Coherent BPSK
  - Coherent QPSK
  - Coherent MSK
  - Coherent FSK
  - DPSK
  - Noncoherent binary FSK

Bit Error Rate

\[
\frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{N_o}}\right)
\]

\[
\frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{2N_o}}\right)
\]

\[
\frac{1}{2} \exp\left(-\frac{E_b}{N_o}\right)
\]

\[
\frac{1}{2} \exp\left(-\frac{E_b}{2N_o}\right)
\]
Comparison of Digital Modulation Schemes

The bit error rate for all the systems decrease monotonically with increasing value of $E_b / N_o$. 