Baseband Data Transmission I

After this lecture, you will be able to

- describe the components of a digital transmission system
  - Information source, transmitter, channel, receiver and destination

- calculate the signaling rate and bit rate of a system

- design the matched filter of a receiver
  - derive the condition for maximum signal-to-noise ratio at the receiver

- determine the error rate
  - Error rate versus received signal energy per bit per hertz of thermal noise
Reference

**Introduction**

**Digital communication system**

- Information source
  - produces a message (or a sequence of symbol) to be transmitted to the destination.
- Example 1
  - Analog signal (voice signal): sampling, quantizing and encoding are used to convert it into digital form
Introduction (1)

- Sampling and quantizing

![Diagram showing quantization noise](image)
**Introduction (2)**

- **encoding**

<table>
<thead>
<tr>
<th>Digits</th>
<th>Binary code</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
</tr>
<tr>
<td>1</td>
<td>001</td>
</tr>
<tr>
<td>2</td>
<td>010</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
</tr>
</tbody>
</table>

**Return-to-zero**
Introduction (3)

- Example 2
  - digital source from a digital computer

Transmitter

- operates on the message to produce a signal suitable for transmission over the channel.
**Channel**
- medium used to transmit the signal from transmitter to the receiver
- Attenuation and delay distortions
- Noise

**Receiver**
- performs the reverse function of the transmitter
  - determine the symbol from the received signal
    - Example: 1 or 0 for a binary system

**Destination**
- the person or device for which the message is intended.
Signaling Rate

Digital message
- An ordered sequence of symbols drawn from an alphabet of finite size $\mu$.
  - Example
    - Binary source: $\mu=2$ for alphabet 0,1 where 0 and 1 are symbols
    - A 4 level signal has 4 symbols in its alphabet such as $\pm 1$, $\pm 3$

Signaling Rate
- The symbols are suitably shaped by a shaping filter into a sequence of signal-elements. Each signal-element has the same duration of $T$ second and is transmitted immediately one after another, so that the signal-element rate (signaling rate) is $1/T$ elements per second (bauds).
Bit Rate

Symbols: 1, 2, 3 and 4

Binary digit: 0 and 1
Bit Rate

- The bit rate is the product of signaling rate and no of bit/symbol.

- Example
  - A 4-level PAM with a signaling rate = 2400 bauds/s.
  - Bit rate (Data rate) = $2400 \times \log_2(4) = 4800$ bits/s (bps)
A basic problem that often arises in the study of communication systems is that of detecting a pulse transmitted over a channel that is corrupted by channel noise.
Matched Filter (2)

- A matched filter is a linear filter designed to provide the maximum signal-to-noise power ratio at its output. This is very often used at the receiver.

\[ x(t) = g(t) + w(t) \quad y(t) = g_o(t) + n(t) \]

- Consider that the filter input \( x(t) \) consists of a pulse signal \( g(t) \) corrupted by additive noise \( w(t) \). It is assumed that the receiver has knowledge of the waveform of the pulse signal \( g(t) \). The source of uncertainty lies in the noise \( w(t) \). The function of receiver is to detect the pulse signal \( g(t) \) in an optimum manner, given the received signal \( x(t) \).
Matched Filter (3)

- The purpose of the circuit is to design an impulse response \( h(t) \) of the filter such that the output signal-to-noise ratio is maximized.

**Signal Power**

Let \( g(f) \) and \( h(f) \) denoted the Fourier Transform of \( g(t) \) and \( h(t) \).

\[
g_0(t) = \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)df
\]

The signal power = \( \left| g_0(t) \right|^2 = \left| \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)df \right|^2 \)
Matched Filter (4)

**Noise Power**

- Since $w(t)$ is white with a power spectral density $\frac{N_0}{2}$, the spectral density function of Noise is

\[ S_N(f) = \frac{N_0}{2} |H(f)|^2 \]

- The noise power $E[n^2(t)] = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$
Matched Filter (5)

S/N Ratio

- Thus the signal to noise ratio become

\[
\eta = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df
\]

\[
\left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi ft) df \right|^2
\]

\[
\int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi ft) df
\]

\[
\eta = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df
\]

\[
\left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi ft) df \right|^2
\]

• (the output is observed at \( t = T \) )
Matched Filter (6)

- Our problem is to find, for a given $G(f)$, the particular form of the transfer function $H(f)$ of the filter that makes $\eta$ at maximum.

**Schwarz’s inequality:**

If $\int_{-\infty}^{\infty} |\phi_1(x)|^2 \, dx < \infty$ and $\int_{-\infty}^{\infty} |\phi_2(x)|^2 \, dx < \infty$, then

$$\left| \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x) \, dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 \, dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 \, dx$$

This equality holds, if and only if, we have $\phi_1(x) = k\phi_2^*(x)$ where $k$ is an arbitrary constant, and $*$ denotes complex conjugation.
Applying the schwarz’s inequality to the numerator of equation (1), we have

\[
\left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi f T} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |G(f)|^2 df
\]

……..(2)

(Note: \( |e^{j2\pi f T}| = 1 \))
Substituting (2) into (1),

The $S/N$ ratio \[ \eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df \]

or

\[ \eta \leq \frac{2E}{N_0} \]

\[ \text{....}(3) \]

where the energy \[ E = \int_{-\infty}^{\infty} |G(f)|^2 df \] is the input signal energy.
Notice that the S/N ratio does not depend on the transfer function $H(f)$ of the filter but only on the signal energy. The optimum value of $H(f)$ is then obtained as

$$H(f) = kG^*(f) \exp(-j2\pi ft)$$
Matched Filter (10)

Taking the inverse Fourier transform of $H(f)$ we have

$$h(t)=k\int_{-\infty}^{\infty} G^*(f) \exp[-j2\pi f (T - t)]df$$

and $G^*(f) = G(-f)$ for real signal $g(t)$

$$h(t)=k\int_{-\infty}^{\infty} G(-f) \exp[-j2\pi f (T - t)]df$$

$$h(t)=kg(T-t) \quad \text{......(4)}$$

Equation (4) shown that the impulse response of the filter is the time-reversed and delayed version of the input signal $g(t)$. "Matched with the input signal"
Matched Filter (11)

Example:

The signal is a rectangular pulse.

\[ g(t) \]

\[ A \]

\[ T \quad t \]

The impulse response of the matched filter has exactly the same waveform as the signal.

\[ h(t) \]

\[ kA \]

\[ T \quad t \]
The output signal of the matched filter has a triangular waveform.
In this special case, the matched filter may be implemented using a circuit known as integrate-and-dump circuit.

\[ \int_{0}^{T} r(t) \]

Sample at \( t = T \)
Realization of the Matched filter (1)

Assuming the output of $y(t) = r(t) \otimes h(t)$

$$= \int_{0}^{t} r(\tau) h(t - \tau) d\tau \quad \ldots (5)$$

Substitute (4) into (5) we have

$$y(t) = \int_{0}^{t} r(\tau) g[T - (t - \tau)] d\tau$$

When $t = T$

$$y(T) = \int_{0}^{T} r(\tau) g(\tau) d\tau$$
Realization of the Matched filter (2)
Error Rate of Binary PAM (1)

Signaling
- Consider a non-return-to-zero (NRZ) signaling (sometime called bipolar). Symbol 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration.

Noise
- The channel noise is modeled as additive white Gaussian noise of zero mean and power spectral density $N_0/2$. In the signaling interval $0 \leq t \leq T_b$, the received signal is

\[ x(t) = \begin{cases} 
+ A + w(t) & \text{symbol 1 was sent} \\
- A + w(t) & \text{symbol 0 was sent} 
\end{cases} \]

- $A$ is the transmitted pulse amplitude
- $T_b$ is the bit duration
Error Rate of Binary PAM (2)

Receiver

- It is assumed that the receiver has prior knowledge of the pulse shape, but not its polarity.
- Given the noisy signal $x(t)$, the receiver is required to make a decision in each signaling interval.
In actual transmission, a decision device is used to determine the received signal. There are two types of error:

- Symbol 1 is chosen when a 0 was actually transmitted
- Symbol 0 is chosen when a 1 was actually transmitted

Case I

- Suppose that a symbol 0 is sent then the received signal is

\[ x(t) = -A + n(t) \]

If the signal is input to a bandlimited low pass filter (matched filter implemented by the integrate-and-dump circuit), the output \( y(t) \) is obtained as:

\[
y(t) = \int_{0}^{t} x(t) dt = -A + \frac{1}{T_b} \int_{0}^{T_b} n(t) dt
\]
As the noise $n(t)$ is white and Gaussian, we may characterize $y(t)$ as follows:

- $y(t)$ is Gaussian distributed with a mean of $-A$
- the variance of $y(t)$ can be shown as $\sigma_y^2 = \frac{N_0}{2T_b}$

(Proof refers to p.254, S. Haykin, Communication Systems)
The Probability density function of a Gaussian distributed signal is given as

\[ f_y(y|0) = \frac{1}{\sqrt{2\pi\sigma_y}} \exp\left(-\frac{(y - \bar{y})^2}{2\sigma_y^2}\right) \]

\[ \therefore f_y(y|0) = \frac{1}{\sqrt{\pi N_0 / T_b}} \exp\left(-\frac{(y + A)^2}{N_0 / T_b}\right) \]

where \( f_y(y|0) \) is the conditional probability density function of the random variable \( y \), given that 0 was sent.
Let $p_{10}$ denote the conditional probability of error, given that symbol 0 was sent.

- This probability is defined by the shaded area under the curve of $f_y(y|0)$ from the threshold $\lambda$ to infinity, which corresponds to the range of values assumed by $y$ for a decision in favor of symbol 1.
Error Rate of Binary PAM (7)

The probability of error, conditional on sending symbol 0 is defined by

\[ P_{10} = P(y > \lambda | \text{Symbol 0 was sent}) \]

\[ = \int_{\lambda}^{\infty} f_y(y|0) \, dy \]

\[ = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y + A)^2}{N_0 / T_b}\right) dy \]
• Assuming that symbols 0 and 1 occur with equal probability, i.e.
  \[ P_0 = P_1 = \frac{1}{2} \]

• If no noise, the output at the matched filter will be – \( A \) for symbol 0 and \( A \) for symbol 1. The threshold \( \lambda \) is set to be 0.
Define a new variable $z = \frac{y + A}{\sqrt{N_o / T_b}}$

and then $dy = \sqrt{\frac{N_0}{T_b}} dz$.

We have

$$P_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b / N_o}}^{\infty} \exp(-z^2) dz$$

where $E_b$ is the transmitted signal energy per bit, defined by $E_b = A^2 T_b$
At this point we find it convenient to introduce the definite integration called **complementary error function**.

\[
\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-z^2)dz
\]

Therefore, the conditional probability of error

\[
P_{10} = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)
\]

( Note: \(\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-z^2)dz\) and \(\text{erfc}(u) = 1 - \text{erf}(u)\) )
In some literature, Q function is used instead of erfc function.

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du
\]

\[
Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad \text{and} \quad \text{erfc}(x) = 2Q(x\sqrt{2})
\]
Case II

Similarly, the conditional probability density function of $Y$ given that symbol 1 was sent, is

$$f_y(y|1) = \frac{1}{\sqrt{\pi N_0 / T_b}} \exp\left(-\frac{(y - A)^2}{N_0 / T_b}\right)$$

$$P_{01} = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y - A)^2}{N_0 / T_b}\right) dy$$
By setting $\lambda = 0$ and putting
\[
\frac{y - A}{\sqrt{N_0 / T_b}} = -z
\]
we find that $P_{01} = P_{10}$

The average probability of symbol error $P_e$ is obtained as
\[
P_e = P_0 P_{10} + P_1 P_{01}
\]

If the probability of 0 and 1 are equal and equal to $\frac{1}{2}$
\[
P_e = \frac{1}{2} \text{erfc}(\sqrt{\frac{E_b}{N_0}})
\]
Error Rate of Binary PAM (13)