Baseband Data Transmission I

After this lecture, you will be able to

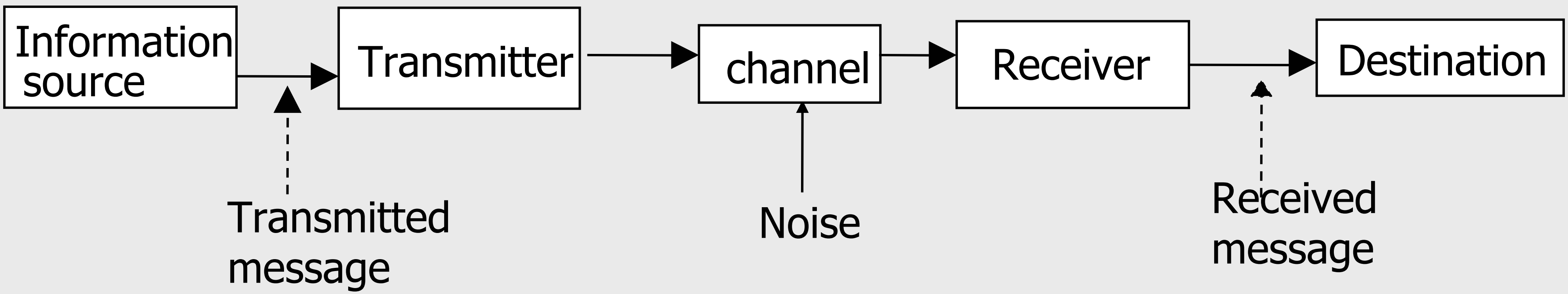
- describe the components of a digital transmission system
  - Information source, transmitter, channel, receiver and destination
- calculate the signaling rate and bit rate of a system
- design the matched filter of a receiver
  - derive the condition for maximum signal-to-noise ratio at the receiver
- determine the error rate
  - Error rate versus received signal energy per bit per hertz of thermal noise

Reference

Reference

– Chapter 4.1 - 4.3, S. Haykin, *Communication Systems*, Wiley.

Digital communication system

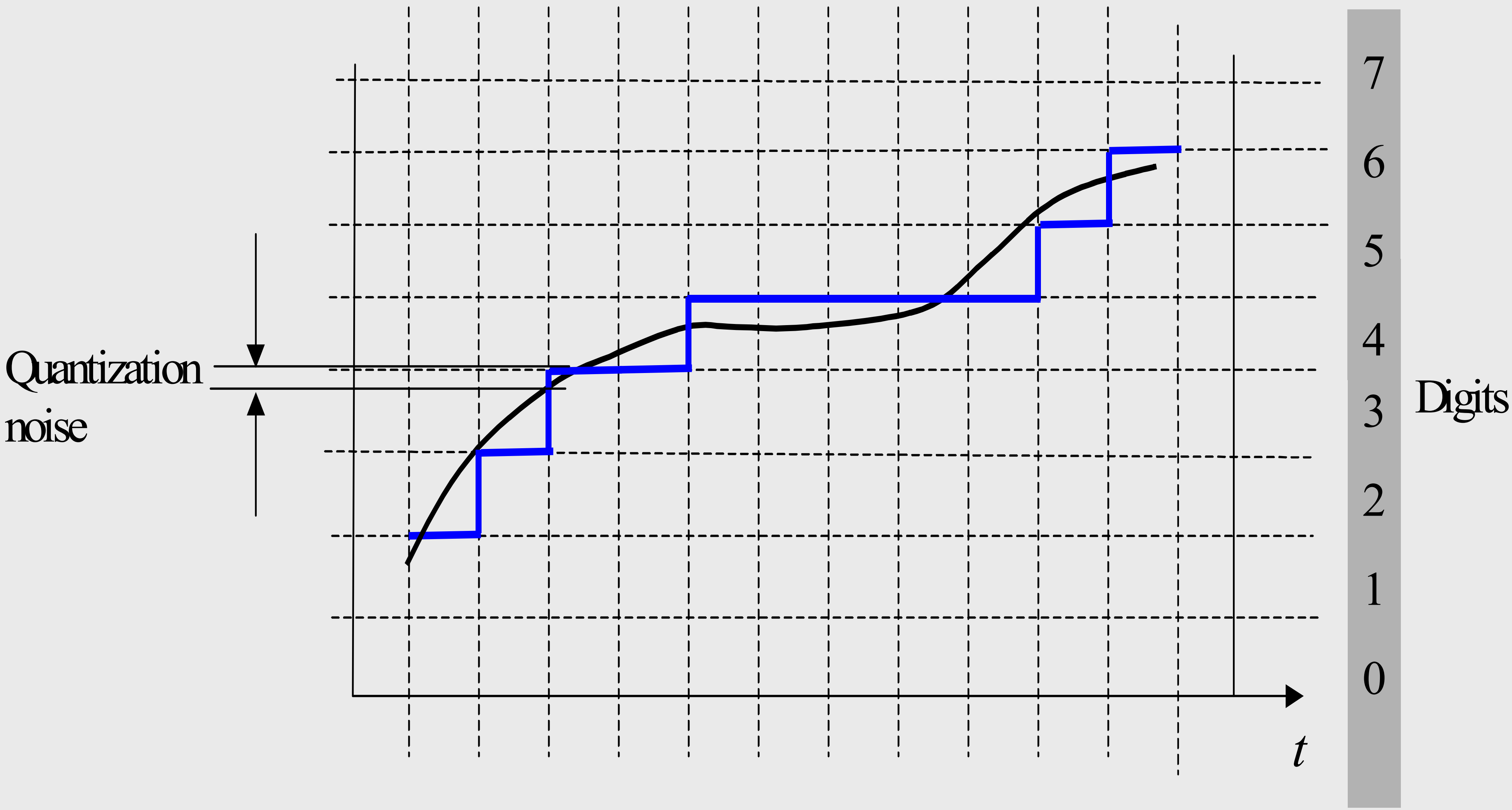


Information source

- produces a message (or a sequence of symbol) to be transmitted to the destination.
- Example 1
  - Analog signal (voice signal): sampling, quantizing and encoding are used to convert it into digital form

Introduction (1)

– Sampling and quantizing

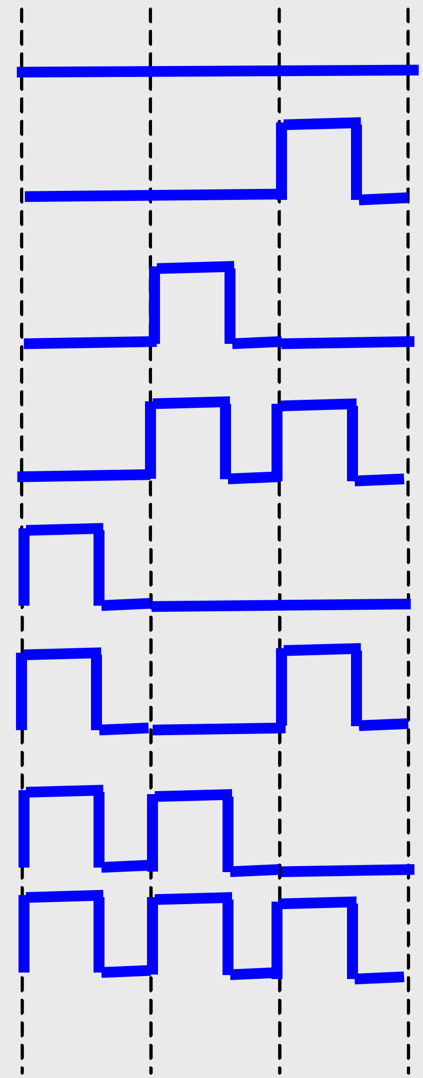


Introduction (2)

– encoding

Digits	Binary code
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Return-to-zero



Introduction (3)

- Example 2
  - digital source from a digital computer

Transmitter

- operates on the message to produce a signal suitable for transmission over the channel.

Introduction (4)

Channel

- medium used to transmit the signal from transmitter to the receiver
- Attenuation and delay distortions
- Noise

Receiver

- performs the reverse function of the transmitter
  - determine the symbol from the received signal
    - Example: 1 or 0 for a binary system

Destination

- the person or device for which the message is intended.

Signaling Rate

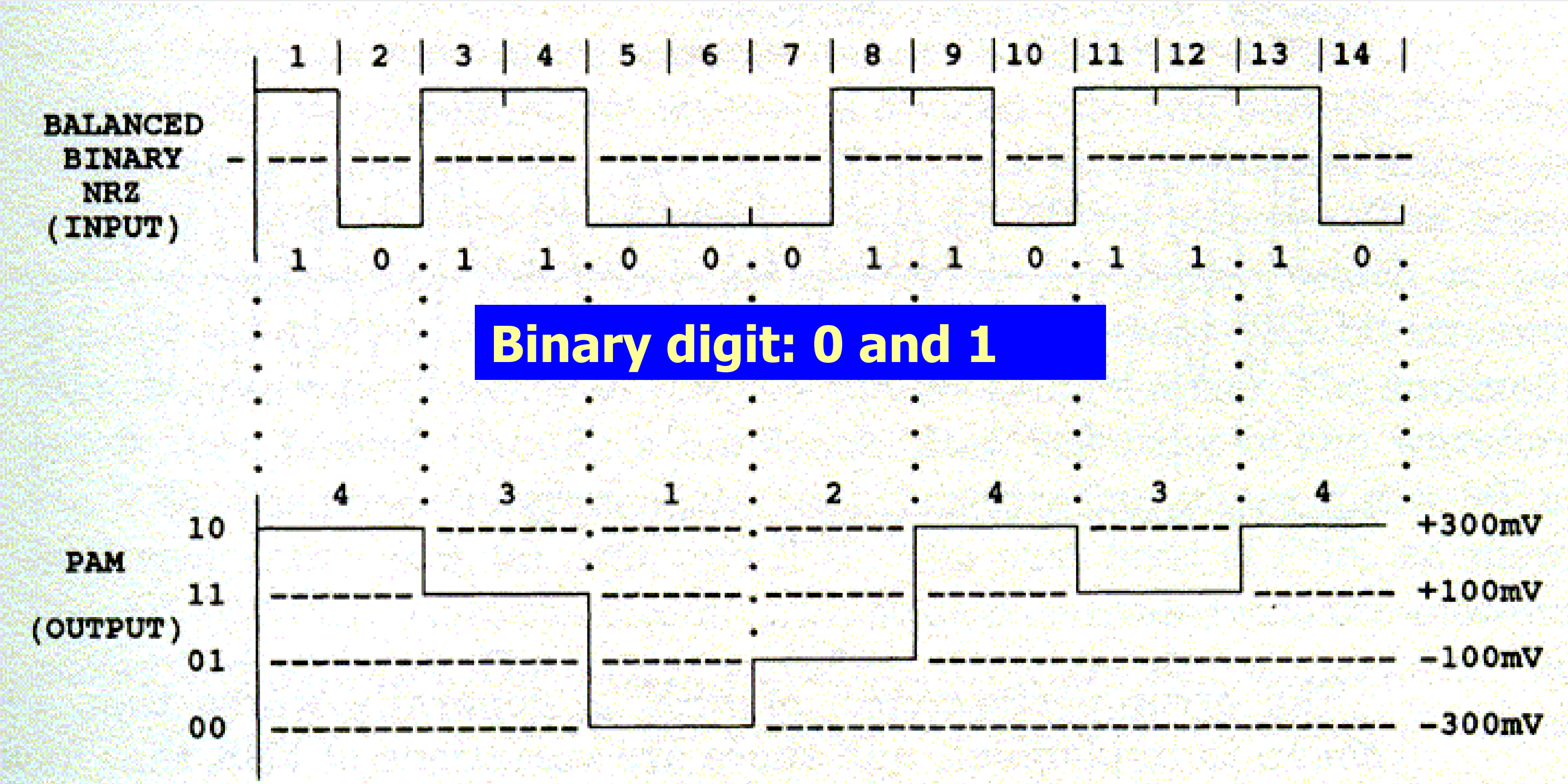
Digital message

– An ordered sequence of symbols drawn from an alphabet of finite size μ.
  - Example
    - Binary source: μ=2 for alphabet 0,1 where 0 and 1 are symbols
    - A 4 level signal has 4 symbols in its alphabet such as ±1, ±3

Signaling Rate

– The symbols are suitably shaped by a shaping filter into a sequence of signal-elements. Each signal-element has the same duration of T second and is transmitted immediately one after another, so that the signal-element rate (signaling rate) is $1/T$ elements per second (bauds).

Bit Rate



Binary digit: 0 and 1

Symbols: 1,2,3 and 4

Bit Rate

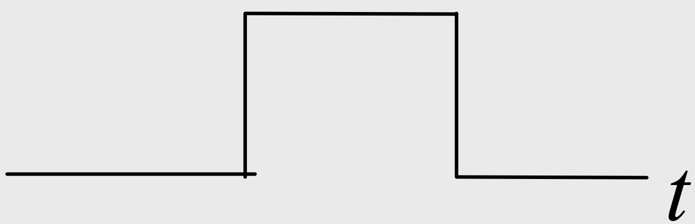
Bit Rate

- The bit rate is the product of signaling rate and no of bit/symbol.

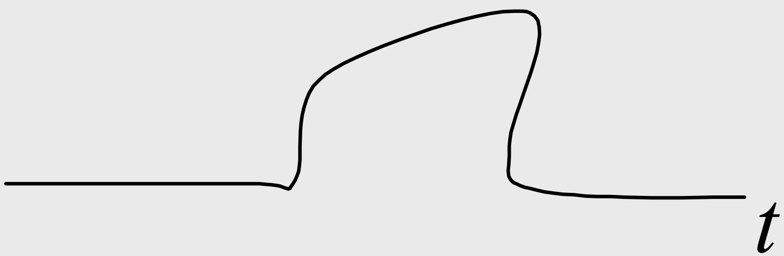
- Example
  - A 4-level PAM with a signaling rate = 2400 bauds/s.
  - Bit rate (Data rate) =2400 X $\log_2(4)$ = 4800 bits/s (bps)

Matched Filter (1)

- A basic problem that often arises in the study of communication systems is that of detecting a pulse transmitted over a channel that is corrupted by channel noise



Square pulse

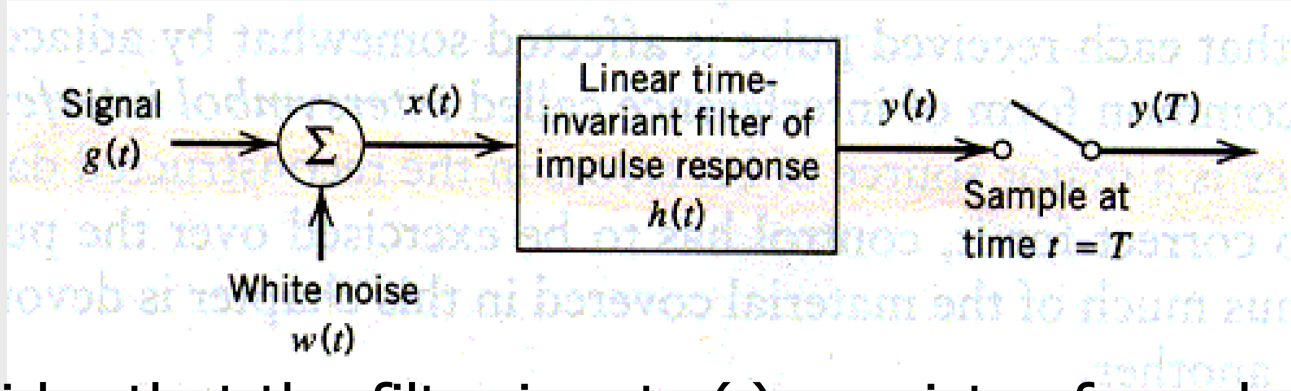


Signal at the receiving end

Matched Filter (2)

- A matched filter is a linear filter designed to provide the <u>maximum signal-to-noise power ratio at its output</u>. This is very often used at the receiver.

$$x(t) = g(t) + w(t) \qquad\qquad y(t) = g_o(t) + n(t)$$



- Consider that the filter input $x(t)$ consists of a pulse signal $g(t)$ corrupted by additive noise $w(t)$. It is assumed that the receiver has knowledge of the waveform of the pulse signal $g(t)$. The source of uncertainty lies in the noise $w(t)$. The function of receiver is to detect the pulse signal $g(t)$ in an optimum manner, given the received signal $x(t)$.

Matched Filter (3)

– The purpose of the circuit is to design an impulse response $h(t)$ of the filter such that the output signal-to-noise ratio is maximized.

Signal Power

Let $g(f)$ and $h(f)$ denoted the Fourier Transform of $g(t)$ and $h(t)$.

$$g_0(t) = \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)df$$

$$\text{The signal power} = \left| \; g_0(t) \; \right|^2 = \left| \; \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)df \; \right|^2$$

Matched Filter (4)

Noise Power

- Since $w(t)$ is white with a power spectral density $\frac{N_0}{2}$, the spectral density function of Noise is

$$S_N(f) = \frac{N_0}{2}\left|H(f)\right|^2$$

- The noise power = $E[n^2(t)] = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df$

Matched Filter (5)

S/N Ratio

– Thus the signal to noise ratio become

$$\eta = \frac{\left| \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)df \right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} \left|H(f)\right|^2 df} \qquad \ldots\ldots.(1)$$

- (the output is observed at $t = T$)

Matched Filter (6)

– Our problem is to find, for a given $G(f)$, the particular form of the transfer function $H(f)$ of the filter that makes η at maximum.

Schwarz's inequality:

If $\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx < \infty$ and $\int_{-\infty}^{\infty} |\phi_2(x)|^2 dx < \infty$,

$$\left| \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx \right|^2 \le \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx$$

This equality holds, if and only if, we have $\phi_1(x) = k\phi_2^*(x)$ where k is an arbitrary constant, and * denotes complex conjugation.

Matched Filter (7)

Applying the schwarz's inequality to the numerator of equation (1), we have

$$\left| \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)df \right|^2 \le \int_{-\infty}^{\infty} \left|H(f)\right|^2 df \int_{-\infty}^{\infty} \left|G(f)\right|^2 df$$

……(2)

(Note: $\left|e^{j2\pi fT}\right| = 1$)

Matched Filter (8)

Substituting (2) into (1) ,

The *S/N* ratio $\eta \le \dfrac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df$

or

$$\eta \le \frac{2E}{N_0} \qquad \ldots\ldots(3)$$

where the energy $E = \int_{-\infty}^{\infty} |G(f)|^2 df$ is the input signal energy

Matched Filter (9)

Notice that the S/N ratio does not depend on the transfer function H(f) of the filter but only on the signal energy. The optimum value of H(f) is then obtained as

$$H(f) = kG^*(f)\exp(-j2\pi fT)$$

Matched Filter (10)

Taking the inverse Fourier transform of $H(f)$ we have

$$h(t)=k\int_{-\infty}^{\infty} G^*(f)\exp[-j2\pi f(T-t)]df$$

and $G^*(f)=G(-f)$ for real signal $g(t)$

$$h(t)=k\int_{-\infty}^{\infty} G(-f)\exp[-j2\pi f(T-t)]df$$

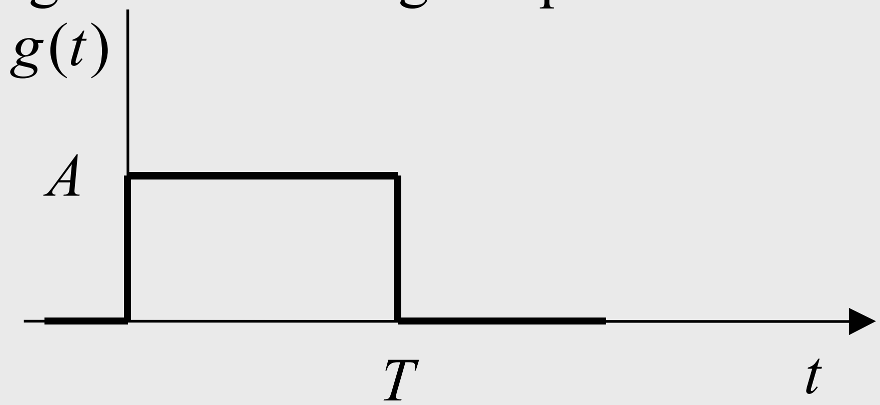
$$h(t)=kg(T\text{-}t) \qquad \ldots\ldots(4)$$

Equation (4) shown that the impulse response of the filter is the time-reversed and delayed version of the input signal $g(t)$. ***"Matched with the input signal"***

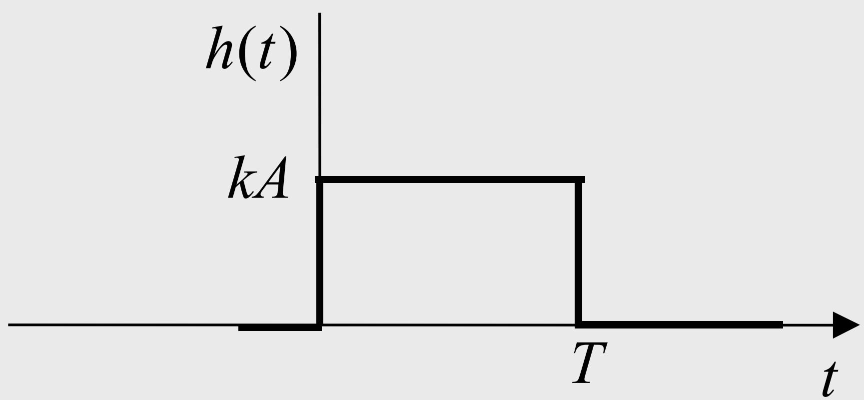
Matched Filter (11)

Example:

The signal is a rectangular pulse.

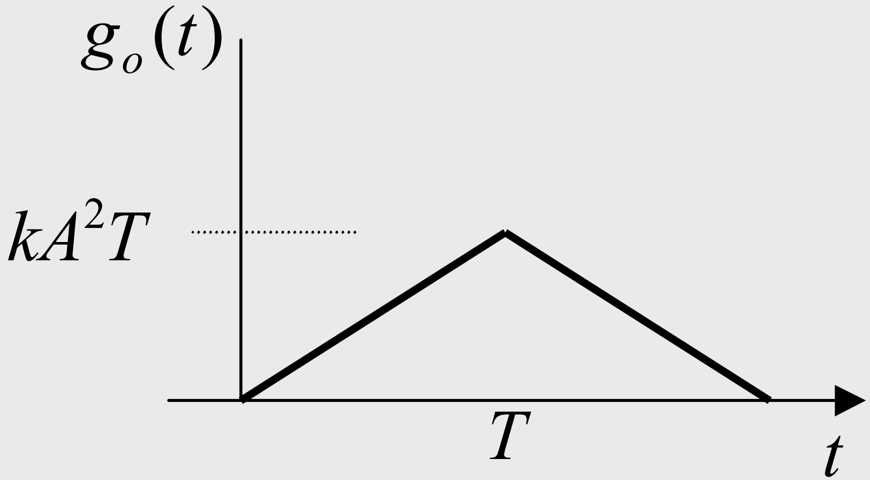


The impulse response of the matched filter has exactly the same waveform as the signal.



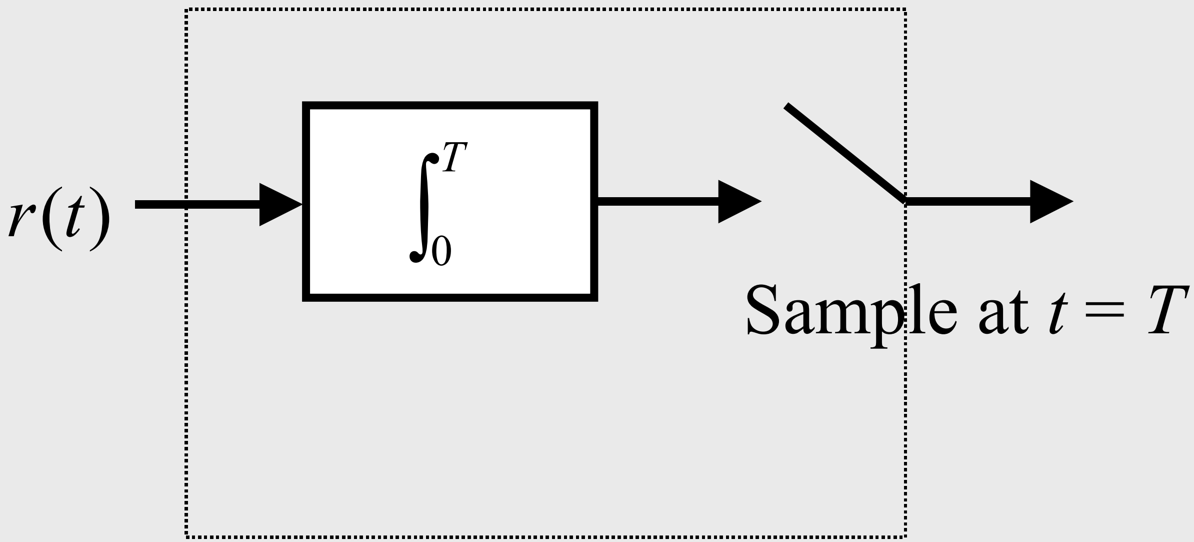
Matched Filter (12)

The output signal of the matched filter has a triangular waveform.



Matched Filter (13)

In this special case, the matched filter may be implemented using a circuit known as integrate-and-dump circuit.



Realization of the Matched filter (1)

Assuming the output of $y(t) = r(t) \otimes h(t)$

$$= \int_0^t r(\tau) h(t-\tau) d\tau \quad ...(5)$$

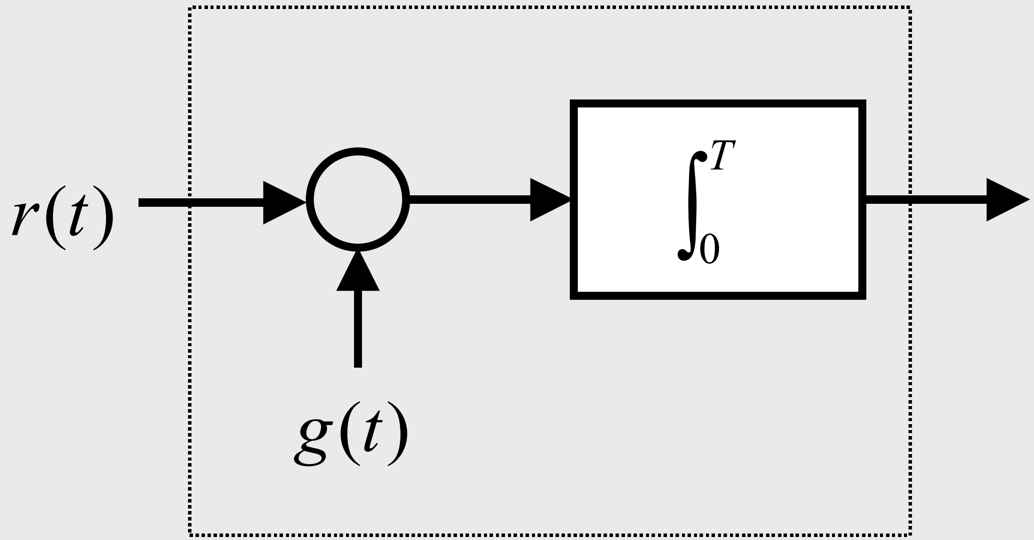
Substitute (4) into (5) we have

$$y(t) = \int_0^t r(\tau) g[T-(t-\tau)] d\tau$$

When $t = T$

$$y(T) = \int_0^T r(\tau) g(\tau) d\tau$$

Realization of the Matched filter (2)



Correlator

Error Rate of Binary PAM (1)

Signaling

- Consider a non-return-to-zero (NRZ) signaling (sometime called bipolar). Symbol 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration.

Noise

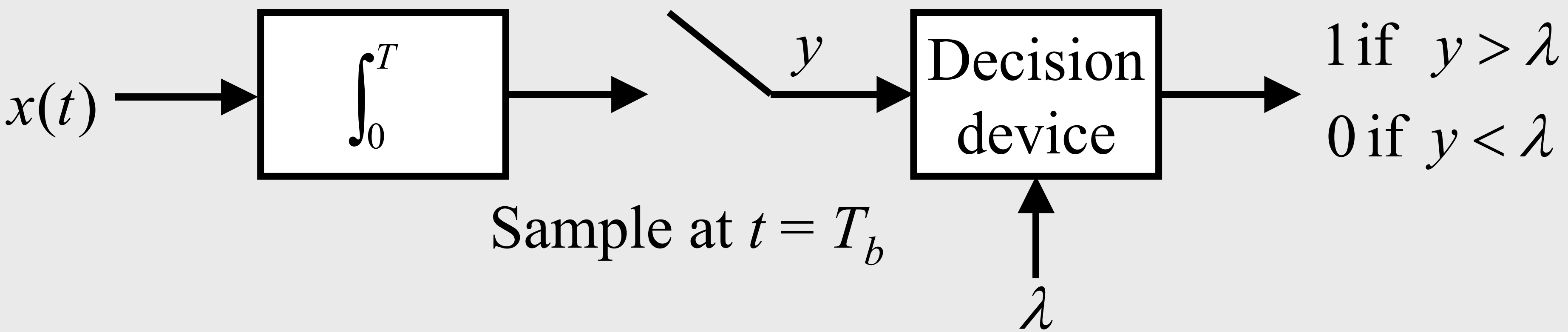
- The channel noise is modeled as additive white Gaussian noise of zero mean and power spectral density $N_o/2$. In the signaling interval $0 \le t \le T_b$, the received signal is

$$x(t) = \begin{cases} +A + w(t) & \text{symbol 1 was sent} \\ -A + w(t) & \text{symbol 0 was sent} \end{cases}$$

  - A is the transmitted pulse amplitude
  - T_b is the bit duration

Error Rate of Binary PAM (2)

Receiver



- It is assumed that the receiver has prior knowledge of the pulse shape, but not its polarity.
- Given the noisy signal $x(t)$, the receiver is required to make a decision in each signaling interval

Error Rate of Binary PAM (3)

- In actual transmission, a decision device is used to determine the received signal. There are two types of error
  - Symbol 1 is chosen when a 0 was actually transmitted
  - Symbol 0 is chosen when a 1 was actually transmitted

Case I

- Suppose that a symbol 0 is sent then the received signal is

$$x(t) = -A + n(t)$$

If the signal is input to a bandlimited low pass filter (matched filter implemented by the integrate-and-dump circuit), the output $y(t)$ is obtained as:

$$y(t) = \int_0^t x(t)dt = -A + \frac{1}{T_b}\int_0^{T_b} n(t)dt$$

Error Rate of Binary PAM (4)

As the noise $n(t)$ is white and Gaussian, we may characterize $y(t)$ as follows:

- $y(t)$ is Gaussian distributed with a mean of –A
- the variance of $y(t)$ can be shown as $\sigma_y^2 = \dfrac{N_0}{2T_b}$

(Proof refers to p.254, *S. Haykin*, *Communication Systems*)

Error Rate of Binary PAM (5)

The Probability density function of a Gaussian distributed signal is given as

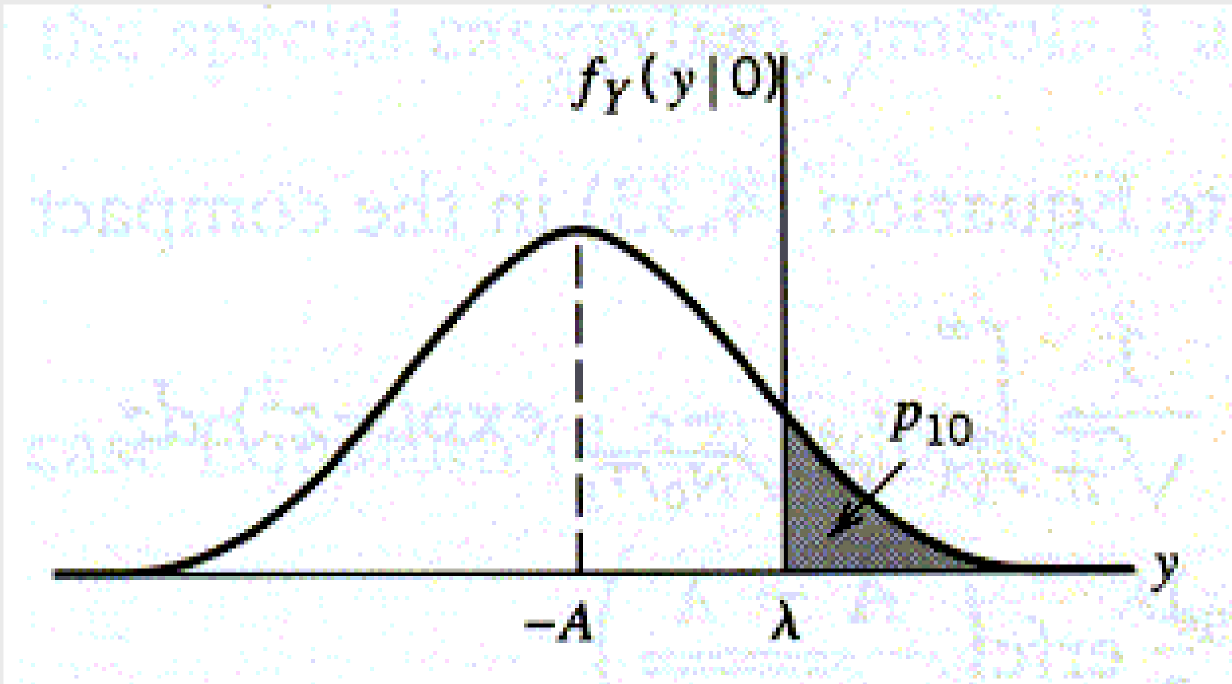
$$f_y(y|0) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp(-\frac{(y-\overline{y})^2}{2\sigma_y^2})$$

$$\therefore f_y(y|0) = \frac{1}{\sqrt{\pi N_0 / T_b}} \exp(-\frac{(y+A)^2}{N_0 / T_b})$$

where $f_y(y|0)$ is the conditional probability density function of the random variable y, given that 0 was sent

Error Rate of Binary PAM (6)

- Let p_{10} denote the conditional probability of error, given that symbol 0 was sent
  - This probability is defined by the shaded area under the curve of $f_y(y\,|\,0)$ from the threshold λ to infinity, which corresponds to the range of values assumed by y for a decision in favor of symbol 1



Error Rate of Binary PAM (7)

The probability of error, conditional on sending symbol 0 is defined by

$$P_{10} = P(y > \lambda \,|\, \text{Symbol 0 was sent})$$

$$= \int_{\lambda}^{\infty} f_y(y|0)\,dy$$

$$= \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y+A)^2}{N_0 / T_b}\right) dy$$

Error Rate of Binary PAM (8)

- Assuming that symbols 0 and 1 occur with equal probability, i.e.
  $$P_0 = P_1 = 1/2$$
- If no noise, the output at the matched filter will be $-A$ for symbol 0 and A for symbol 1. The threshold λ is set to be 0.

Error Rate of Binary PAM (9)

Define a new variable $z = \dfrac{y + A}{\sqrt{N_o / T_b}}$

and then $dy = \sqrt{\dfrac{N_0}{T_b}}\, dz.$

We have $$P_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b / N_o}}^{\infty} \exp(-z^2)\, dz$$

where E_b is the transmitted signal energy per bit, defined by $E_b = A^2 T_b$

Error Rate of Binary PAM (9)

At this point we find it convenient to introduce the definite integration called complementary error function.

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\,dz$$

Therefore, the conditional probability of error

$$P_{10} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

(Note: $\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^{u} \exp(-z^2)\,dz$ and $\mathrm{erfc}(u) = 1 - \mathrm{erf}(u)$)

Error Rate of Binary PAM (10)

In some literature, Q function is used instead of erfc function.

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-\frac{u^2}{2}) du$$

$$Q(x) = \frac{1}{2} \text{erfc}(\frac{x}{\sqrt{2}}) \text{ and } \text{erfc}(x) = 2Q(x\sqrt{2})$$

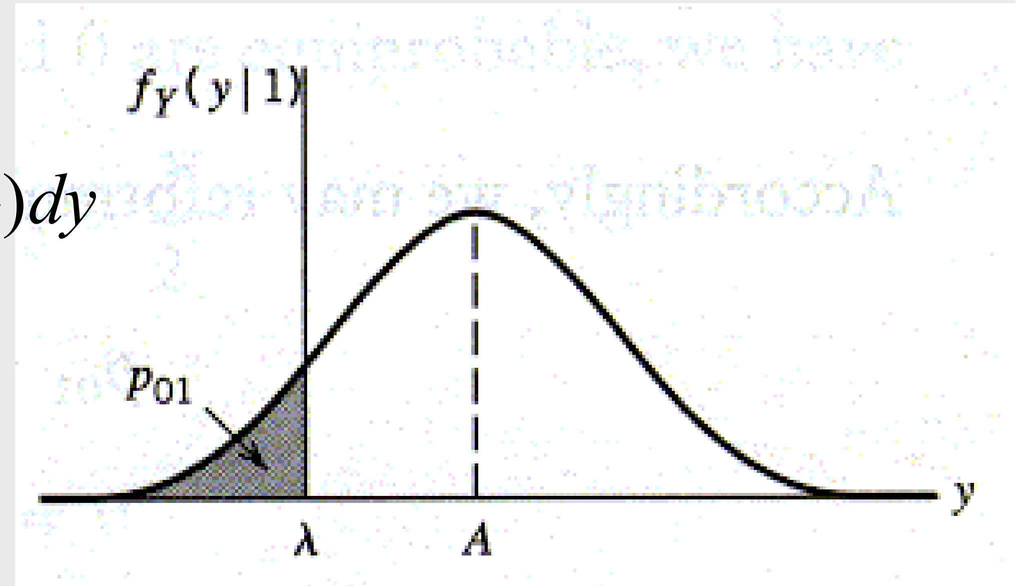
Error Rate of Binary PAM (11)

Case II

Similary, the conditional probability density function of Y given that symbol 1 was sent, is

$$f_y(y|1) = \frac{1}{\sqrt{\pi N_0 / T_b}} \exp\left(-\frac{(y-A)^2}{N_0 / T_b}\right)$$

$$P_{01} = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y-A)^2}{N_0 / T_b}\right) dy$$



Error Rate of Binary PAM (12)

By setting $\lambda = 0$ and putting

$$\frac{y - A}{\sqrt{N_0 / T_b}} = -z$$

we find that $P_{01} = P_{10}$

The average probability of symbol error P_e is obtained as

$$P_e = P_0 P_{10} + P_1 P_{01}$$

If the probability of 0 and 1 are equal and equal to ½

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Error Rate of Binary PAM (13)