16

Chaos in Power Electronics: An Overview

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Abstract

Power electronics is rich in nonlinear dynamics. Its operation is characterized by cyclic switching of circuit topologies, which gives rise to a variety of nonlinear behavior. This chapter provides an overview of the chaotic dynamics and bifurcation scenarios observed in power electronics circuits, emphasizing the salient features of the circuit operation and the modelling strategies. This chapter covers the modelling approaches, analysis methods, and a classification of the common types of bifurcations observed in power electronics.

¹Financial support for C.K. Tse to undertake this work has been provided by the Hong Kong Research Grants Council under Grant PolyU-5131/99E.

16.1 Introduction

Power electronics is a discipline spawned by real-life applications in industrial, commercial, residential and aerospace environments. Much of the development of the field of power electronics evolves around some immediate needs for solving specific power conversion problems. In the past three decades, power electronics has gone through intense development in many aspects of technology [1], including power devices, control methods, circuit design, computer-aided analysis, passive components, packaging techniques, etc. Moreover, the principal focus in power electronics, as reflected from topics discussed in some key power electronics conferences [2, 3], is to fulfill the functional requirements of the application for which the concerned circuits are used. Like many areas of engineering, power electronics is mainly motivated by practical applications, and it often turns out that a particular circuit topology or system implementation has found widespread applications long before it is thoroughly analyzed and most of its subtleties uncovered. For instance, the widespread application of a simple switching converter may date back to more than three decades ago. However, good analytical models allowing better understanding and systematic circuit design were only developed in late 1970's [4], and in-depth analytical and modelling work is still being actively pursued today. Nonlinear phenomena, despite being commonly found in power electronics circuits, have only received formal treatments in very recent years.

In this chapter, we review some of the nonlinear phenomena observed in typical power electronics systems. Our aim is to give an account of methods and techniques that can be applied to study the many “strange” phenomena previously observed in power electronics. Simple dc/dc converters are used as representative examples to illustrate the modelling approaches that are capable of retaining the essential qualitative properties. Such properties, peculiar to switching and non-smooth systems, are systematically analyzed in terms of possible bifurcation scenarios and nonlinear phenomena. Our intention is to provide an overview of the recent research efforts in the study of nonlinear phenomena in power electronics, and to summarize the essential analytical approaches that can be used to perform a systematic classification of the nonlinear behavior of power electronics systems. Since we wish to limit ourselves to a concise overview of the main results, we will ignore the analytical details and simply refer the readers to the main references listed at the end of the chapter. The rest of the chapter is outlined as follows. We begin in Sec. 16.2 with a brief overview of power electronics circuits, followed by a review of the classical methods of analysis in Sec. 16.3 and a discussion of the benefits of studying chaotic dynamics in power electronics in Sec. 16.4. A quick tour of some recent findings is presented in Sec. 16.5. In Secs. 16.6 and 16.7, we present a detailed discussion of the modelling methods for characterizing nonlinear phenomena

FIGURE 16.1
Simple dc/dc converters. (a) Buck converter; (b) buck-boost converter; (c) boost converter.

such as bifurcation and chaos in power electronics.

16.2 Power Electronics Circuits: A Brief Overview

Regardless of its specific function, a power electronics circuit operates by toggling its topology among a set of linear or nonlinear circuit topologies, under the control of a feedback system. As such, they can be regarded as piecewise-switched circuits.

For example, in simple dc/dc converters, such as the ones shown in Fig. 16.1, an inductor is ‘switched’ between the input and the output through an appropriate switching element (labelled as S in the figure). The way in which the inductor is switched determines the output voltage level and transient behavior. Usually, a semiconductor switch and a diode are used to implement the said switching, and through the use of a feedback control circuit, the relative durations of the various switching intervals can be continuously adjusted. Such feedback action effectively controls the dynamics and steady-state behavior of the circuit. Thus, both the circuit topology and the control method determines the dynamical behavior of a power electronics circuit.

16.2.1 Simple DC/DC converters

Many power electronics converters are constructed on the basis of the three simple converters shown in Fig. 16.1. In a typical period-1 operation, the switch S and the diode D are turned on and off in a cyclic and complementary fashion, under the command of a pulse-width modulator. When the switch S is closed (the diode D is open), the inductor current ramps up. When the switch S is open (the diode D closed), the inductor current ramps down. The duty ratio, defined as the fraction of a repetition period during which S is closed, is continuously controlled by a feedback circuit that aims to maintain the output voltage at a fixed level even under input and load variations. Two typical feedback arrangements are shown in the next subsection.

The behavior of a dc/dc converter is greatly influenced by its operating mode. Typically, we can distinguish two different modes of operation, namely, *continuous conduction mode* (CCM) and *discontinuous conduction mode* (DCM). In CCM, the inductor current is maintained non-zero throughout the entire repetition cycle. This happens when the inductance L is relatively large, or the load current demand is relatively high. Moreover, in DCM, the inductor current is zero for an interval of time within a cycle. This happens when the inductance L is relatively small, or the load current demand is low, causing the inductor current to fall to zero as the inductor is being discharged. During the interval of zero inductor current, both switch S and diode D are open.

For both CCM and DCM, the output voltage has a fixed relationship with the input voltage, as determined by the duty ratio. A steady-state relationship can be easily found from a volt-time balance consideration of the inductor. Table 16.1 shows the steady-state output voltage expressions for stable period-1 operation [5]. Here, we denote by δ_s the steady-state duty ratio, and by T the repetition period.

As we will see in this chapter, although the period-1 stable operation is the preferred operation for most industrial applications, it represents only one particular operating regime. Because of the existence of many possible operating regimes, it would be of practical importance to have a thorough understanding of what determines the behavior of the circuit so as to guarantee a desired operation or to avoid an undesirable one.

16.2.2 Typical control strategies

Most dc/dc converters are designed to deliver a regulated output voltage. The control of dc/dc converters usually takes on two approaches, namely voltage feedback control and current-programmed control, also known as *voltage-mode* and *current-mode* control, respectively [6]. In voltage-mode control, the output voltage is compared with a reference to generate a control signal which drives

FIGURE 16.2

Typical control approaches for dc/dc converters: (a) Voltage-mode control; (b) current-mode control.

Converter type	CCM	DCM
Buck	$V_o = \delta_s V_{in}$	$V_o = 2V_{in} \left(1 + \sqrt{1 + \frac{8L}{RT\delta_s^2}} \right)^{-1}$
Buck-boost	$V_o = \frac{\delta_s}{1 - \delta_s} V_{in}$	$V_o = V_{in} \delta_s \sqrt{\frac{RT}{2L}}$
Boost	$V_o = \frac{1}{1 - \delta_s} V_{in}$	$V_o = \frac{V_{in}}{2} \left(1 + \sqrt{1 + \frac{2\delta_s^2 L}{RT}} \right)$

TABLE 16.1

Steady-state output voltage expressions for stable period-1 operation [5]. δ_s denotes the steady-state duty ratio and T denotes the switching period.

the pulse-width modulator via some typical feedback compensation configuration. For current-mode control, an inner current loop is used in addition to the voltage feedback loop, the aim of which is to force the peak inductor current to follow a reference signal which is derived from the output voltage feedback loop. The result of current-mode control is a faster response. This kind of control is mainly applied to boost and buck-boost converters which suffer from an undesirable non-minimum phase response [5, 6]. The simplified schematics are shown in Fig. 16.2.

16.3 Conventional Treatments

As mentioned previously, power electronics circuits are essentially piecewise-switched circuits [7]. The number of possible circuit topologies is usually fixed, and the switching is done in a cyclic manner (but not necessarily periodically because of the feedback action). This results in a nonlinear time-varying operating mode, which naturally demands the use of nonlinear methods for analysis and design. Indeed, researchers and engineers who work in this field are always dealing with nonlinear problems and have attempted to explore methods not normally used in other circuit design areas, e.g., state-space averaging [4], phase-plane trajectory analysis [8], Lyapunov based control [9], Volterra series approximation [10], etc. However, in order to expedite the design of power electronics systems, “adequate” simplifying models are imperative. In the process of deriving models, accuracy is often traded off for simplicity for many good practical reasons. Since closed-loop stability and transient responses are basic design concerns in practical power electronics systems, models that can permit the direct application of conventional frequency-domain approaches will present obvious advantages. Thus, much research in modelling power electron-

ics circuits has been directed towards the derivation of a linear model that is appropriate to a frequency-domain analysis; the limited validity being the price to pay. (The fact that most engineers are trained to use linear methods is also a strong motivation for developing linearized models.) For example, the averaging approach [4], one of the most widely adopted modelling approaches for switching converters, initially yields simple nonlinear models that contain no time-varying parameters and hence can be used more conveniently for analysis and design. Essentially, an averaged model discards the switching details and focuses only on the envelope of the dynamical motion. This is well suited to characterize power electronics circuits in the low-frequency domain.

In practice, moreover, such so-called “averaged” models are often linearized to yield linear time-invariant models that can be directly studied in a standard Laplace transform domain or frequency domain, facilitating design of control loops and evaluation of transient responses in ways that are familiar to practitioners.

16.4 Bifurcations and Chaos in Power Electronics

Power electronics engineers frequently encounter such phenomena as subharmonic oscillations, jumps, quasi-periodic operations, sudden broadening of power spectra, bifurcations and chaos, despite not knowing what causes them [11, 12]. Most power supply engineers would have experienced bifurcation phenomena and chaos in switching regulators when some parameters (e.g., input voltage and feedback gain) are varied, but usually do not examine the phenomena in detail. The usual reaction of the engineers is to avoid these phenomena by adjusting component values and parameters, often through some trial-and-error procedure. Thus, the phenomena remain somewhat mysterious and rarely examined in a formal manner.

What has been said so far may be regarded as a kind of stereotypical development in application-driven disciplines. Indeed, if linear models can be profitably used in design (and as long as the restricted validity of the models does not adversely compromise the design integrity), there seems to be no immediate needs for investigating such nonlinear phenomena as chaos and bifurcation. However, as the field of power electronics gains maturity and as the demand for better functionality, reliability and performance of power electronics systems increases, in-depth analysis into nonlinear behavior and phenomena becomes justifiable and even mandatory. On the one hand, the study of nonlinear phenomena offers the opportunity of rationalizing the commonly observed behavior. Thus, knowing how and when chaos occurs, for instance, will certainly help avoid it, if “avoiding it” is what the engineers want. On the other hand, many previously unused nonlinear operating regimes may be profitably

exploited for useful engineering applications, provided that such operations are thoroughly understood. For these reasons, the study of bifurcations and chaos in power electronics has recently attracted much attention from both the power electronics and the circuits and systems communities.

In the next section, we present a chronological survey of the recent findings in the identification, analysis and modelling of nonlinear phenomena in power electronics circuits and systems.

16.5 A Survey of Research Findings

The occurrence of bifurcations and chaos in power electronics was first reported in the literature in the late eighties by Hamill *et al.* [13, 14]. Experimental observations regarding boundedness, chattering and chaos were also made by Krein and Bass [15] back in 1990. Although these early reports did not contain any rigorous analysis, they seriously pointed out the importance of studying the complex behavior of power electronics and its likely benefits for practical design. Since then, much interest has been aroused in the power electronics and circuits research communities in pursuing formal studies of the complex phenomena commonly observed in power electronics.

In 1990, Hamill *et al.* [16] presented a paper at the IEEE Power Electronics Specialists Conference, reporting an attempt to study chaos in a simple buck converter which became a subject of intensive research in the following decade (and still is!). Using an implicit iterative map, the occurrence of period-doublings, subharmonics and chaos in a simple buck converter was demonstrated by numerical analysis, PSPICE simulation and laboratory measurement. The derivation of a closed-form iterative map for the boost converter under a current-mode control scheme was presented later by the same group of researchers [17]. This closed-form iterative map allowed the analysis and classification of bifurcations and structural instabilities of this simple converter. Since then, a number of authors have contributed to the identification of bifurcation patterns and strange attractors in a wider class of circuits and devices of relevance in power electronics. Some key publications are summarized below. See also [18–20] for alternative reviews.

The occurrence of period-doubling cascades for a simple dc/dc converter operating in DCM was reported in 1994 by Tse [21, 22]. By modelling the dc/dc converter operating in DCM as a first-order iterative map, the onset of period-doubling bifurcations can be located analytically. The idea is based on evaluating the Jacobian of the iterative map about the fixed point corresponding to the solution undergoing the period-doubling, determining the condition for which a period-doubling bifurcation occurs (i.e., Jacobian equals -1). Simulations and laboratory measurements have confirmed the findings. Formal

FIGURE 16.3

Bifurcation diagrams from a current-mode controlled boost converter with $L = 1.5$ mH, $R = 40 \Omega$ and $T = 100 \mu\text{s}$. For (a), $T/CR = 0.125$, and for (b), $T/CR = 0.625$.

theoretical studies of conditions for the occurrence of period-doubling cascades in discontinuous-mode dc/dc converter were reported subsequently in [23, 24]. Further work on the bifurcation behavior of the buck converter was reported by Chakrabarty *et al.* [25] who specifically studied the bifurcation behavior under variation of a range of circuit parameters including storage inductance, load resistance, output capacitance, etc. In 1996, Fossas and Olivar [26] presented a detailed analytical description of the buck converter dynamics, identifying the topology of its chaotic attractor and studying the regions associated with different system evolutions. Various possible types of operation of a simple voltage-feedback pulse-width-modulated buck converter were also investigated through the so-called stroboscopic map obtained by periodically sampling the system states. This method will be discussed later in this chapter.

The bifurcation behavior of dc/dc converters under current-mode control has been studied by a number of authors. Deane [27] first reported on the route to chaos in a current-mode controlled boost converter. Chan and Tse [28] studied various types of routes to chaos and their dependence upon the choice of bifurcation parameters. Fig. 16.3 shows two bifurcation diagrams numerically obtained from a current-mode controlled boost converter, with output current level being the bifurcation parameter. Fig. 23.1 shows a series of trajectories as the current level increases. In 1995, the study of bifurcation phenomena was extended to a fourth-order Ćuk dc/dc converter under a current-mode control scheme [29]. The four dimensional system is represented by an implicit fourth-order iterative map, from which routes to chaos are identified numerically.

As can be seen from Fig. 16.3 (a) for the case of the boost converter, the bifurcation behavior contains transitions where a “sudden jump from periodic solutions to chaos” is observed. These transitions cannot be explained in terms

FIGURE 16.4

Trajectories from a current-mode controlled boost converter. (a) Stable period-1 operation; (b) stable period-2 operation; (c) stable period-4 operation; and (d) chaotic operation.

of standard bifurcations such as “period-doubling” and “saddle-node”. In fact, as studied by Banerjee *et al.* [30, 31] and Di Bernardo [32], these transitions are due to a novel class of bifurcation known as “grazing” or “border-collision” bifurcation, which is unique to switched dynamical systems [33–35].

Since most power electronics circuits are non-autonomous systems driven by fixed-period clock signals, the study of the dynamics can be effectively carried out using appropriate discrete-time maps. In addition to the aforementioned stroboscopic maps (which will be discussed in the next section), Di Bernardo *et al.* [24] studied alternative sampling schemes and their applications to the study of bifurcation and chaos in power electronics [32]. It has been found that non-uniform sampling can be used to derive discrete-time maps (termed A-switching maps) which can be used to effectively characterize the occurrence of bifurcations and chaos in both autonomous and non-autonomous systems. These maps turned out to be particularly useful for the investigation of non-smooth bifurcation in power electronics circuits and systems. Also, the occurrence of periodic chattering was explained in terms of sliding solutions [36].

When external clocks are absent and the system is “free-running”, for example, under a hysteretic control scheme, the system is autonomous and does not have a fixed switching period. Such free-running converters were indeed extremely common in the old days when fixed-period integrated-circuit controllers were not available. For this type of autonomous converters, chaos cannot occur if the system order is below three. A representative example is the free-running Ćuk converter which has been shown by Tse *et al.* [37] to exhibit Hopf bifurcation and chaos.

Power electronics circuits other than dc/dc converters have also been examined in recent years. Dobson *et al.* [38] reported “switching time bifurcation” of diode and thyristor circuits. Such bifurcation manifests as jumps in the switching times. Bifurcation phenomena from induction motor drives were reported separately by Kuroe [39] and Nagy *et al.* [40]. Finally, some attempts have been made to study higher order parallel-connected systems of converters which are becoming popular design choice for high current applications [41].

16.6 Modelling Strategies

In Sec. 16.3, we discussed the conventional “averaging” method for modelling power electronics circuits. In essence, averaging retains the low-frequency properties while ignoring the detailed dynamics within a switching cycle. Usually, the validity of averaged models is only restricted to the low-frequency range up to an order of magnitude below the switching frequency. For this reason, averaged models become inadequate when the aim is to explore nonlinear phenomena that may appear across a wide spectrum of frequencies. Nevertheless, averaging techniques can be useful to analyze those bifurcation phenomena which are confined to the low-frequency range. For instance, in a free-running Ćuk converter, Hopf bifurcation that gives birth to a limit cycle consisting of many switching periods (low-frequency behavior) can be clearly observed from a suitable averaged model [37].

An effective approach for modelling power electronics circuits with a high degree of exactness is to use appropriate discrete-time maps obtained by uniform or non-uniform sampling of the system states. Essentially the aim is to derive an iterative function that expresses the state variables at one sampling instant in terms of those at an earlier sampling instant. In what follows, the basic concepts of discrete-time maps and how they can be used to explore nonlinear phenomena in power electronics are explained.

16.6.1 Discrete-time maps

As is often the case in nonlinear dynamical systems, the analysis of complex phenomena that are of relevance to engineering requires “adequate” models. As mentioned previously, discrete-time maps obtained by suitable sampling of the system dynamics can be extremely useful for characterizing the occurrence of bifurcations and chaos in power electronics circuits. For the sake of clarity, we distinguish two different classes of maps, namely *Poincaré maps* and *normal-form maps*. The former is useful for describing the global dynamics of the system under investigation from one sampling instant to the next. The latter, which is valid locally to a bifurcation point, is an invaluable tool for classifying the system behavior following the onset of a bifurcation. In what follows we focus on the derivation of appropriate Poincaré maps for dc/dc converters. The use of normal-form maps for classification will be left to the next section.

Several kinds of Poincaré maps have been defined for the analysis of nonlinear phenomena in power electronics circuits. For convenience, we use a voltage-controlled buck converter as an representative example to illustrate the derivation of the most commonly used maps, namely, the *stroboscopic* map, the *S-switching* (synchronous switching) map, and the *A-switching* (asynchronous switching) map.

Specifically the buck converter can be described by a piecewise smooth system of the form:

$$\begin{aligned}\frac{di(t)}{dt} &= -\frac{1}{L}v(t) + \frac{\delta(t)}{L}E, \\ \frac{dv(t)}{dt} &= \frac{1}{C}i(t) - \frac{1}{RC}v(t),\end{aligned}\tag{16.6.1}$$

where $i(t)$ is the inductor current, $v(t)$ the capacitor voltage, E a constant input voltage, and $\delta(t)$ a modulated signal which is 0 whenever a linear combination of the system states, $v_c(t) = g_i i(t) + g_v v(t)$, is greater than an appropriately generated ramp signal of period T , $v_r(t) = \gamma + \eta(t \bmod T)$, and is 1 otherwise (i.e., $\delta(t)$ is 0 when the switch is OFF, and is 1 when the switch is ON).

Without loss of generality, let us assume that the converter begins its evolution at a time instant equal to an integral multiple of the modulating period, and that the converter assumes $\delta(t) = 1$ initially. For brevity, we refer to the phase with $\delta(t) = 1$ as *phase 1*, and to that with $\delta(t) = 0$ as *phase 2*. Thus, the converter stays in phase 1 until the next commutation is reached, and so on.

In general, discrete-time maps can be categorized into the following three classes (see Fig. 16.5).

- *Stroboscopic map*: obtained by sampling the system states *periodically* at time instants which are multiples of the period of the ramp signal v_r .

(stroboscopic instants). Note that the system states are sampled at each stroboscopic instant irrespectively of whether the system configuration switches or not at that instant.

- *S-switching map (synchronous switching)*: obtained by sampling the system states at those stroboscopic time instants when the system commutes from phase 2 to phase 1.
- *A-switching map (asynchronous switching)*: obtained by sampling the system states at time instants within each ramp cycle when the system commutes from phase 1 to phase 2.

In order to derive the relevant maps, we introduce the following two simplifying assumptions. These assumptions can be later removed.

1. No more than one commutation takes place during each period of the modulating signal.
2. The commutation from phase 2 to phase 1 can only take place at time instants which are multiples of the ramp cycle T .

Note that the second assumption is always true when the converter is under current-mode control, but is not so for voltage-mode control.

16.6.2 Derivation of stroboscopic and S-switching maps

The stroboscopic map is the most widely used type of discrete-time maps for modelling dc/dc converters. It can be obtained by sampling the system dynamics every T seconds, say, at the beginning of each ramp cycle. To avoid confusing with other types of maps, we use suffix k as the counting index for this map, as shown in Fig. 16.5. Applying a simple iterative procedure to the solutions of the state equations for phases 1 and 2, the stroboscopic map can be obtained as

$$x_{k+1} = N_2(1 - \delta_k)N_1(\delta_k)x_k + [N_2(1 - \delta_k)M_1(\delta_k) + M_2(1 - \delta_k)]E, \quad (16.6.2)$$

where x_k denotes $x(kT)$, δ_k is the duty ratio in the k th period, and the solution to the state equation in each phase is given by $x(\delta) = N_i(\delta) + M_i(\delta)E$ with

$$N_i(\delta) = e^{A_i\delta T} \quad (16.6.3)$$

$$M_i(\delta) = A_i^{-1}(e^{A_i\delta T} - I)B_i. \quad (16.6.4)$$

Moreover, under certain drastic control conditions, the converter may stay in phase 1 or phase 2 for the entire period. In such cases (normally called skipped cycles), one must assume $\delta_k = 1$ or $\delta_k = 0$, as appropriate.

FIGURE 16.5

Two iterations of the stroboscopic, S-switching and A-switching maps under voltage-mode control.

As mentioned above, at time instants multiple of T , the stroboscopic map does not discriminate between instants at which a switching occurs and those characterized by skipped cycles (stroboscopic instants where no switching occurs). In order to take into account this situation, the S-switching map can be used. Essentially the S-switching map is obtained by sampling the state vector at time instants multiple of T at which a switching occurs. For clarity, these samples will be denoted by suffix n . Specifically, the S-switching map can be written as

$$x_{n+1} = N_2(1 - \delta_n + \sigma_2)N_1(\delta_n + \sigma_1)x_n + [N_2(1 - \delta_n + \sigma_2)M_1(\delta_n + \sigma_1) + M_2(1 - \delta_n + \sigma_2)]E, \quad (16.6.5)$$

where σ_1 and σ_2 are the numbers of skipped cycles in phases 1 and 2, respectively, between the time instants t_n and t_{n+1} . Analytically the stroboscopic

and the S-switching maps are identical when $\sigma_1 = 0$ and $\sigma_2 = 0$.

In the open-loop case, i.e., $g^T = (0, 0)$, the stroboscopic map is linear. However, it becomes nonlinear when a closed-loop control is activated. This is because the duty ratio δ_n depends on the system states as a consequence of introducing the feedback control. Moreover, the construction of the map under closed-loop condition usually requires solving δ_n from a control equation, which is not necessarily in closed form. In fact, with the exception of current-mode controlled boost and buck-boost converters with zero inductance's series resistance, the control equation involving δ_n is transcendental and can only be solved numerically. Therefore, under closed-loop condition, the stroboscopic and the S-switching maps cannot be written in closed form for most cases.

16.6.3 Derivation of A-switching maps

The A-switching map is obtained when the state vector is asynchronously sampled at switching times internal to the modulating period, as shown in Fig. 16.5. Specifically we can readily write down the A-switching map by noting that the converter enters phase 2 after an A-switching, i.e.,

$$\begin{aligned} x_{m+1} = & N_1(\delta_{m+1} + \sigma_1)N_2(1 - \delta_m + \sigma_2)x_m \\ & + [N_1(\delta_{m+1} + \sigma_1)M_2(1 - \delta_m + \sigma_2) + M_1(\delta_{m+1} + \sigma_1)]E. \end{aligned} \quad (16.6.6)$$

In order to construct the A-switching map under a closed-loop condition, the variables δ_{m+1} and δ_m must be eliminated from (16.6.6). To do this, we assume that the following conditions are defined by the control law as the conditions for an A-switching:

$$g^T x_m = \alpha + \beta \delta_m T, \quad (16.6.7)$$

$$g^T x_{m+1} = \alpha + \beta \delta_{m+1} T. \quad (16.6.8)$$

Let us assume that $\beta \neq 0$ for simplicity. This assumption will be later removed. By computing δ_m from (16.6.7) and δ_{m+1} from (16.6.8), and putting them in (16.6.6), we obtain the following A-switching map for the closed-loop system:

$$\begin{aligned} x_{m+1} = & N_1(\gamma^T x_{m+1} - a + \sigma_1)N_2(1 - \gamma^T x_m + a + \sigma_2)x_m \\ & + [N_1(\gamma^T x_{m+1} - a + \sigma_1)M_2(1 - \gamma^T x_m + a + \sigma_2) \\ & + M_1(\gamma^T x_{m+1} - a + \sigma_1)]E, \end{aligned} \quad (16.6.9)$$

where $\gamma^T = g^T/(\beta T)$ and $a = \alpha/(\beta T)$. Note that (16.6.9) has a closed form, although it is an implicit map.

As detailed in [19], explicit functional forms can be used by considering appropriate simplified maps. These can be obtained by using a small-signal approximation, i.e., by Taylor-expanding the exponential functions which characterize the maps introduced above.

All the maps introduced above can be used to obtain analytical conditions for the existence of given periodic orbits and standard bifurcations such as period-doubling and saddle-node. Many reports of standard bifurcations, as surveyed earlier, have effectively employed the stroboscopic and S-switching maps [18–20]. Moreover, the A-switching maps are particularly useful for the analysis of multi-switching behavior, and can be constructed more conveniently for operations where skipped cycles are frequent such as when the converter is operating in a chaotic regime.

16.7 Analysis and Classification of Non-smooth Bifurcations

The literature already abounds with methods of analysis and classification of standard bifurcations like period-doubling and saddle-node [42]. In the following we focus on the non-smooth bifurcations, which are particularly relevant to power electronics.

16.7.1 System formulation

As discussed earlier, because of their switching nature, power electronics systems are often modelled by sets of ordinary differential equations (ODEs) or maps which are intrinsically piecewise-smooth (PWS). Specifically, whenever the circuit under investigation switches to a different configuration (for example, from ON to OFF, or vice versa), the vector field of the corresponding model changes from one functional form to another. Thus, mathematically, power electronics circuits can be modelled by dynamical systems of the form:

$$\dot{x} = \begin{cases} F_1(x, \mu, t), & \text{if } x \in S_1 \\ F_2(x, \mu, t), & \text{if } x \in S_2 \\ \dots & \\ F_k(x, \mu, t), & \text{if } x \in S_k \end{cases} \quad (16.7.10)$$

where $F_i : R^{n+p+1} \mapsto R^n$, $i = 1, \dots, k$, is smooth in each of the phase-space regions, S_i , and μ is a system parameter. Also, the system switches from one functional form F_i to another F_j , whenever the system states move from region S_i to region S_j .

Along the boundaries between different regions, say $\Phi_{i,j}$, the system states can have different degrees of discontinuity. In particular, according to the

properties of the system along its discontinuity boundaries, we can identify three main classes of PWS dynamical systems:

- Systems with discontinuous states (jumps);
- Systems with a discontinuous vector field, i.e., the first derivative of the system states is discontinuous across the phase-space boundaries;
- Systems with a discontinuous Jacobian (i.e. $F_{ix} \neq F_{jx}$), i.e., the second derivative of the system states is discontinuous across the phase-space boundaries.

Moreover, the discontinuity boundaries between different phase-space regions can themselves be smooth or piecewise smooth.

For example, consider the dc/dc buck converter depicted in Fig. 16.1 (a) and described by (16.6.1). Suppose that the ramp signal v_r defines a discontinuity boundary, Φ , which is itself discontinuous and divides the phase space into two separate regions associated with the ON and OFF configurations. In this case, the first derivative of the system states varies discontinuously as the boundary Φ is crossed [36].

16.7.2 Bifurcation possibilities

Power electronics systems can exhibit standard bifurcations such as period-doubling or saddle-node. Such bifurcations are indeed frequently observed both analytically and experimentally, as surveyed earlier in Sec. 16.5. Nevertheless, some of the most common dynamical transitions observed in power electronics circuits, such as the “sudden jump to chaos” mentioned earlier, cannot be explained in terms of standard bifurcations [43–45]. In fact, power electronics systems, being *switched dynamical systems*, are known to exhibit an interesting class of bifurcations which cannot be observed in their smooth counterparts. Specifically, for switched dynamical systems, a dramatic change of the system behavior is usually observed when a part of the system trajectory hits tangentially one of the boundaries between different regions in phase space. When this occurs, the system is said to undergo a *grazing* bifurcation which is also known as C-bifurcation in the Russian literature [46–49].

For example, in the case of the buck converter, such an event corresponds to the feedback signal “grazing” the tip of the ramp signal at a stroboscopic point. As studied by Banerjee *et al.* [31] and shown further by Di Bernardo *et al.* [50], it is possible to analyze the system behavior associated with “grazing” by considering an appropriate piecewise linear *normal-form* map of the form:

$$x_{m+1} \mapsto \begin{cases} A_1 x_m + B\mu, & \text{if the system is in phase 1} \\ A_2 x_m + B\mu, & \text{if the system is in phase 2} \end{cases} \quad (16.7.11)$$

FIGURE 16.6
Derivation of normal-form map near a grazing bifurcation.

where A_1, A_2, B are the linear system matrices valid for describing the local behavior near the point of grazing, and μ is a system parameter. In fact, when the system undergoes a grazing bifurcation, a fixed point of such a normal-form map crosses transversally some boundary in the phase space (see Fig. 16.6). This is also called *border-collision bifurcation* whose occurrence in one-dimensional and two-dimensional maps was first reported in the western literature by Nusse *et al.* [33,34].

It is worth noting that such normal-form maps are not always piecewise linear. In fact, it can be shown [51] that the form of the normal-form map at a grazing depends on the discontinuity of the system vector field, and is piecewise linear only if the discontinuity boundary between the ON and OFF zones is itself discontinuous such as in the cases of many power electronics circuits. Thus, knowing the form of the normal-form map associated with a border collision becomes important when the aim is to predict the dynamical behavior of the system following such a bifurcation. In the next subsection, we describe a classification method to identify the scenario following a border collision.

16.7.3 Classification

We begin with a brief description of the derivation of the normal-form map associated with a grazing or border-collision bifurcation. Referring to Fig. 16.6, suppose that as the value of some parameter μ increases, a periodic orbit of system (16.7.10), say L_0 , becomes tangent (grazing) to the switching hyperplane, Φ_0 , when $\mu = \hat{\mu}$. The grazing limit cycle L_0 is associated with a fixed point, say M_0 , on the Poincaré section D . As the parameter μ is varied, such a fixed point moves from M^* to M^{**} , i.e., from a fixed point associated with an orbit which does not cross the boundary to one corresponding to a solution which crosses the boundary. Linearizing the system flow about each of these fixed points, we can then obtain a piecewise linear map of the form (16.7.11). This approximate map is valid if the switching hyperplane Φ_0 is itself non-smooth [51].

An effective method for classifying and predicting the dynamical scenarios following a border-collision bifurcation is given in Di Bernardo *et al.* [52]. The available results (up to now) for the n -dimensional case are summarized as follows. Let σ_1^+ and σ_2^+ be the numbers of eigenvalues, respectively, of A_1 and A_2 in (16.7.11), which are greater than 1. Likewise, let σ_1^- and σ_2^- be the numbers of eigenvalues that are less than -1 . Specifically, a periodic orbit undergoing a border collision will

- smoothly change into one containing an additional section on the other side of the switching hyperplane, if

$$\sigma_1^+ + \sigma_2^+ \text{ is even;} \quad (16.7.12)$$

- suddenly disappear after touching the switching hyperplane, if

$$\sigma_1^+ + \sigma_2^+ \text{ is odd;} \quad (16.7.13)$$

- undergo a period-doubling, if

$$\sigma_1^- + \sigma_2^- \text{ is odd.} \quad (16.7.14)$$

The above three elementary conditions can be used to describe bifurcation scenarios of various degrees of complexity [52, 53], which cover the sudden transition from a periodic orbit to a chaotic attractor. Note that complete classification is available only for two-dimensional systems [54]. At the time of writing, no complete classification for the general n -dimensional case has been reported.

Finally, it is worth mentioning that power electronics systems can exhibit a peculiar type of solution, termed *sliding*, which lies within the system discontinuity set. Intuitively, this can be seen as associated with an infinite number of switchings between different phase-space regions which keep the trajectory

on the discontinuity boundary. The presence of sliding can give rise to the formation of so-called sliding orbits, i.e., periodic solutions characterized by sections of sliding motion (or chattering). These solutions can play an important role in organizing the dynamics of a given power electronics circuit [36]. Research is still on-going in identifying a novel class of bifurcations, called *sliding bifurcations*, which involve interactions between the system trajectories and discontinuity sets where sliding motion is possible [56].

16.8 Current Status and Future Work

Research in nonlinear phenomena of power electronics may be said to have gone through its first phase of development. Most of the work reported so far has focused on identifying the phenomena and explaining them in the language of the nonlinear dynamics literature. The past decade of research may have served a two-fold purpose. First, to the engineers, the commonly observed “strange” phenomena (e.g., chaos and bifurcation) become topics that can be scientifically approached, rather than just being “bad” laboratory observations not relevant to the main technical interest. Second, to the chaos and system theorists, the proliferation of publication in this area has demonstrated the rich dynamics of power electronics, offering a new area for theoretical study [47,51,52,54,57,58].

It seems that identification work will continue to be an important area of investigation. This is because power electronics emphasizes reliability and predictability, and it is imperative to understand the system behavior as thoroughly as possible and under all kinds of operating conditions. Knowing when and how a certain bifurcation occurs, for example, will automatically mean knowing how to avoid it. Furthermore, power electronics is an emerging discipline; new circuits and applications are created every day. The lack of general solutions for nonlinear problems makes it necessary for each application to be studied separately and the associated nonlinear phenomena identified independently.

Future research will inevitably move toward any profitable exploitation of the nonlinear properties of power electronics. As a start, some applications of chaotic power electronics systems and related theory have been identified, for instance, in the control of electromagnetic interference by “spreading” the noise spectrum [59,60], in the application of “targeting” orbits with less iterations (i.e., directing trajectories to certain orbits in as little time as possible) [61], and in the stabilization of periodic operations [62].

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