Convergence Analysis of the Unscented Kalman Filter for Filtering Noisy Chaotic Signals

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***Abstract*— The unscented Kalman filter (UKF) has recently been proposed for filtering noisy chaotic signals. Though computationally advantageous, the UKF has not been thoroughly analyzed in terms of its convergence property. In this paper, non-periodic oscillatory behavior of the UKF when used to filter chaotic signals is reported. We show both theoretically and experimentally that the gain of the UKF may oscillate aperiodically. More precisely, when applied to periodic signals generated from nonlinear systems, the Kalman gain and error covariance of the UKF converge to zero. However, when the system being considered is chaotic, the Kalman gain either converges to a fixed point with a magnitude larger than zero or oscillates aperiodically.**

I. Introduction

A number of noise reduction methods for cleaning noisy chaotic time series have been proposed [1], [2]. These methods are mainly applied to delay-embedded noisy scalar time series in a reconstruction space based on the embedding theorem [3], and to observed noisy time series based on some filtering techniques to obtain an optimal estimation of the original dynamical systems [4]–[7]. The most commonly used tool for filtering noisy chaotic signals is the extended Kalman filter (EKF) algorithm, which makes use of the first-order Taylor series approximation of the nonlinear function. This method, however, has two fundamental deficiencies, namely, high computational complexity due to the need for computing the Jacobian matrices and low estimation precision [8].

Recently, a new type of filters, known as the unscented Kalman filter (UKF), has been proposed for noise cleaning applications [9], [10]. The fundamental difference between EKF and UKF lies in the way in which Gaussian random variables (GRV) are represented in the process of propagating through the system dynamics. Basically, the UKF captures the posterior mean and covariance of GRV accurately to the third order (in terms of Taylor series expansion) for any form of nonlinearity, whereas the EKF only achieves first-order accuracy. Moreover, since no explicit Jacobian or Hession calculations are necessary in the UKF algorithm, the computational complexity of UKF is comparable to EKF. Very recently, Feng and Xie [11] applied the UKF algorithm to filter noisy chaotic signals and equalize blind channels for chaos-based communication systems. The results indicate that the UKF algorithm outperforms conventional adaptive filter algorithms including the EKF algorithm.

Up till now, the convergence of the UKF algorithm has not been analyzed. In this paper, we evaluate the performance of UKF in filtering chaotic signals generated from 1-dimensional discrete-time dynamical systems, which basically include most behaviors and characteristics in many multidimensional systems, and are widely used in signal processing and communications with chaos.

II. Theoretical Analysis

Consider a nonlinear dynamical system, which is given by

$$\boldsymbol{x}_n = \boldsymbol{f}(\boldsymbol{x}_{n-1}), \tag{1}$$

where $f : R^N \rightarrow R^N$ is a smooth function, and the measurement equation is given by

$$\boldsymbol{y}_n = \boldsymbol{x}_n + \boldsymbol{v}_n, \tag{2}$$

where $\boldsymbol{v}_n$ is a zero-mean white Gaussian noise process, $E(\boldsymbol{v}_j\boldsymbol{v}_n) = \mathbf{R}\delta_{jn} > 0$ ($E(\cdot)$ denotes expectation), and δ_{jn} is the Kronecker delta function, $\mathbf{R}$ is a matrix with a suitable dimensionality. When the global Lyapunov exponent of (1) is positive, the system is chaotic. We will employ the UKF algorithm developed in [9], [10] to filter the noisy chaotic time series, i.e., to estimate the state $\boldsymbol{x}_n$ from $\boldsymbol{y}_n$.

Assume that the statistics of random variable $\boldsymbol{x}$ (dimension L) has mean $\bar{\boldsymbol{x}}$ and covariance $\mathbf{P}_{\boldsymbol{x}}$. To calculate the statistics of $\boldsymbol{y}$, we form a matrix χ of $2L+1$ *sigma* vector χ_i according to the following:

$$\chi_0 = \bar{\boldsymbol{x}} \tag{3}$$

$$\chi_i = \bar{\boldsymbol{x}} + (\sqrt{(L+\lambda)\mathbf{P}_{\boldsymbol{x}}})_i,\ i = 1, 2, \cdots, L \tag{4}$$

$$\chi_i = \bar{\boldsymbol{x}} - (\sqrt{(L+\lambda)\mathbf{P}_{\boldsymbol{x}}})_{i-L},\ i = L+1, L+2, \cdots, 2L \tag{5}$$

where $\lambda = \alpha^2(L+\kappa) - L$ is a scaling parameter. The constant α determines the spread of the sigma points around $\bar{x}$ and is usually set to a small positive value ($0.0001 < \alpha < 1$). The constant κ is a secondary scaling parameter which is usually set to $\kappa = 0$ (for parameter estimation, $\kappa = 3 - L$ [9]). Also, $(\sqrt{(L+\lambda)\mathbf{P}_{\boldsymbol{x}}})_i \stackrel{\text{def}}{=} \boldsymbol{e}_i$ is the vector of the ith column of the matrix square root. These sigma vectors are propagated through the nonlinear function

$$\boldsymbol{z}_i = \boldsymbol{\varphi}(\chi_i),\ i = 0, \cdots, 2L. \tag{6}$$

The mean and the covariance of $\boldsymbol{z}$ can be approximated by using the weighted sample mean and covariance of the posterior sigma points, i.e.,

$$\bar{\boldsymbol{z}} = \sum_{i=0}^{2L} w_i^{(m)} \boldsymbol{z}_i \quad (7)$$

$$\mathbf{P}_{\boldsymbol{zz}} = \sum_{i=0}^{2L} w_i^{(c)} (\boldsymbol{z} - \bar{\boldsymbol{z}})(\boldsymbol{z} - \bar{\boldsymbol{z}})^T \quad (8)$$

where weights w_i are given by

$$w_0^m = \frac{\lambda}{L+\lambda} \quad (9)$$

$$w_0^c = \frac{\lambda}{L+\lambda} + (1 - \alpha^2 + \beta) \quad (10)$$

$$w_i^m = w_i^c = \frac{1}{2(L+\lambda)}, \ (i = 1, 2, \cdots, 2L) \quad (11)$$

where β is used to incorporate prior knowledge of the distribution of $\boldsymbol{x}$ (for Gaussion distributions, $\beta = 2$ is optimal) [9]. Now the algorithm can be generalized as follows:

Time update:

$$\chi_{n|n-1} = \boldsymbol{f}(\chi_{n-1}) \quad (12)$$

$$\hat{\boldsymbol{x}}_{n|n-1} = \sum_{i=0}^{2L} w_i^m \chi_{i,n|n-1} \quad (13)$$

$$\mathbf{P}_{n|n-1} = \sum_{i=0}^{2L} w_i^c (\chi_{i,n|n-1} - \hat{\boldsymbol{x}}_{n|n-1})(\chi_{i,n|n-1} - \hat{\boldsymbol{x}}_{n|n-1})^T \quad (14)$$

$$\mathbf{P}_{\boldsymbol{x}_n \boldsymbol{y}_n} = \sum_{i=0}^{2L} w_i^c (\chi_{i,n|n-1} - \hat{\boldsymbol{x}}_{n|n-1})(\boldsymbol{y}_{i,n|n-1} - \hat{\boldsymbol{y}}_{n|n-1})^T \quad (15)$$

where $\chi_{n|n-1}$, $\hat{\boldsymbol{x}}_{n|n-1}$ and $\mathbf{P}_{n|n-1}$ is the predicted estimation for χ_{n-1}, $\boldsymbol{x}_{n-1}$ and $\mathbf{P}_{n-1}$ respectively. $\mathbf{P}_{\boldsymbol{x}_n \boldsymbol{y}_n}$ is the covariance matrix of $\boldsymbol{x}$ and $\boldsymbol{y}$ at instant n.

Measurement update:

$$\mathbf{K}_n = \mathbf{P}_{\boldsymbol{x}_n \boldsymbol{y}_n} (\mathbf{P}_{\boldsymbol{y}_n \boldsymbol{y}_n})^{-1} \quad (16)$$

$$\hat{\boldsymbol{x}}_n = \hat{\boldsymbol{x}}_{n|n-1} + \mathbf{K}_n (\boldsymbol{y}_n - \hat{\boldsymbol{y}}_{n|n-1}) \quad (17)$$

$$\mathbf{P}_n = \mathbf{P}_{n|n-1} - \mathbf{K}_n \mathbf{P}_{\boldsymbol{y}_n \boldsymbol{y}_n} \mathbf{K}_n^T \quad (18)$$

where $\mathbf{P}_{\boldsymbol{y}_n \boldsymbol{y}_n}$ is the covariance matrix of $\boldsymbol{y}$ at instant n, $\hat{\boldsymbol{y}}_{n|n-1}$ is the estimation of the observed signal, and $\mathbf{K}_n$ is the Kalman gain vector at time instant t. Using (2), (8), (14) and (15), we can get

$$\mathbf{P}_{n|n-1} = \mathbf{P}_{\boldsymbol{x}_n \boldsymbol{y}_n} = \mathbf{P}_{\boldsymbol{y}_n \boldsymbol{y}_n} - \mathbf{R}_n, \quad (19)$$

which then gives

$$\mathbf{P}_n = \mathbf{K}_n \mathbf{R}_n \,, \quad (20)$$

where $\mathbf{R}_n$ is the covariance matrix of the measurement noise. Furthermore, $\mathbf{P}_{n|n-1}$ can be calculated according to the UKF algorithm as

$$\mathbf{P}_{n|n-1} = \mathbf{F}_{n-1}^T \mathbf{P}_{n-1} \mathbf{F}_{n-1} + \frac{1}{4}(\beta - \alpha^2) \cdot \left[(\mathbf{H}_{n-1}\mathbf{P}_{n-1})^T (\mathbf{H}_{n-1}\mathbf{P}_{n-1})\right] \quad (21)$$

where $\mathbf{F}_{n-1}$ and $\mathbf{H}_{n-1}$ are the Jacobian and Hession matrices of (1), respectively. Being of 1-dimension, (21) can be simplified as

$$P_{n|n-1} = F_{n-1}^2 P_n + \frac{1}{4}(\beta - \alpha^2) H_{n-1}^2 P_{n-1}^2. \quad (22)$$

By using (20), (21) and (22), we can calculate the Kalman gain as

$$k_n = \frac{(F_{n-1}^2 + \frac{1}{4}(\beta - \alpha^2) H_{n-1}^2 P_{n-1}^2) k_{n-1}}{(F_{n-1}^2 + \frac{1}{4}(\beta - \alpha^2) H_{n-1}^2 P_{n-1}) k_{n-1} + 1}. \quad (23)$$

Note that both P_n and k_n can be used to measure the filter's performance. However, since they are linearly dependent, as shown in (20), it suffices to analyze the behavior of any one of them. For simplicity, we focus on k_n.

In general, four types of behavior can be identified in a nonlinear dynamical system:

- stable fixed point;
- periodic motion;
- quasiperiodic motion;
- chaotic state.

In a 1-dimensional dissipative system, a quasi-periodic state corresponds to a zero Lyapunov exponent, and it is a critical state with zero measure of parameters. It is therefore not of interest to our present study. Here, we consider two classes of dynamical behavior, namely, chaotic and periodic, as the stable fixed point can be regarded as a special periodic state. Our analysis for the UKF algorithm used to filter a chaotic system is summarized in the following theorems. In particular, we consider three types of nonlinear dynamical systems which determine the behavior of UKF.

- **Type 1**: $F_{n-1}^2$is independent of x_{n-1}.
- **Type 2**: F_{n-1}^2 is dependent upon x_{n-1}, but H_{n-1}^2 is independent of x_{n-1}.
- **Type 3**: H_{n-1}^2 is dependent upon x_{n-1}.

Theorem 1: For Type 1 systems, the Kalman gain k_n given in (23) converges to zero when the systems are periodic, and converges to a non-zero fixed point when the systems are chaotic.

Proof: As F_{n-1}^2 is independent of x_{n-1}, F_{n-1}^2 can be regarded as a constant ξ. Thus, (23) becomes

$$k_n = \frac{\xi k_{n-1}}{\xi k_{n-1} + 1} \overset{\text{def}}{=} \phi(k_{n-1}, \xi), \quad (24)$$

which includes two fixed points, i.e.,

$$k^{(1)} = 0, \ k^{(2)} = \frac{\xi - 1}{\xi}. \quad (25)$$

The stability of the two fixed points can be determined by the derivative of ϕ at the corresponding fixed points. If the derivative has a magnitude larger than one, the fixed point is unstable, and the Kalman gain cannot converge. Otherwise, the fixed point is stable. By evaluating the derivative of (24) at the two fixed points, we have

$$\phi'(k^{(1)}) = \xi, \ \phi'(k^{(2)}) = \frac{1}{\xi}. \quad (26)$$

When $\xi < 1$, $k_n = 0$ is a stable fixed point, and $k_n = \frac{\xi-1}{\xi}$ is an unstable fixed point. Conversely, when $\xi > 1$, $k_n = 0$ is an unstable fixed point, and $k_n = \frac{\xi-1}{\xi}$ is a stable fixed point.

The global Lyapunov exponent of the 1-dimensional system can be written as

$$\zeta = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln |F_n| = \frac{1}{2} \ln(\xi). \tag{27}$$

Thus, when (1) is periodic, $\xi < 1$, and k_n converges to $k^{(1)}$. Moreover, when (1) is chaotic, k_n converges to $k^{(2)}$.

Theorem 2: For Type 2 systems, if $H_{n-1}^2 = q \neq 0$ (q is a constant), the Kalman gain k_n in (23) converges to zero when the systems are periodic, and k_n oscillates aperiodically when the systems are chaotic.

Proof: From (23), we obtain

$$k_n = \frac{[F_{n-1}^2 + \frac{1}{4}q(\beta - \alpha^2)]k_{n-1}}{[F_{n-1}^2 + \frac{1}{4}q(\beta - \alpha^2)]k_{n-1} + 1}. \tag{28}$$

If (1) is chaotic, and the square of the gradient F_n^2 is aperiodic, then $F_{n-1}^2 + \frac{1}{4}q(\beta - \alpha^2)$ is also aperiodic. Hence, the result follows according to Theorem 2 in [12].

If we define $\boldsymbol{e}_{n-1}\boldsymbol{e}_{n-1}^T = P_{n-1}$, then $H_{n-1}\boldsymbol{e}_{n-1} = F_n - F_{n-1}$. By using (23), we can obtain

$$k_n = \frac{[F_{n-1}^2 + \frac{1}{4}(\beta - \alpha^2)(F_n - F_{n-1})^2]k_{n-1}}{[F_{n-1}^2 + \frac{1}{4}(\beta - \alpha^2)(F_n - F_{n-1})^2]k_{n-1} + 1}. \tag{29}$$

Lemma 1: Suppose $G_n = F_{n-1}^2 + \frac{1}{4}(\beta - \alpha^2)(F_n - F_{n-1})^2$. Then, G_n is aperiodic.

Proof: Assuming that G_n has a period of N, without loss of generality, we have

$$G_{n+N} - G_n = 0. \tag{30}$$

Define

$$m_{1,n} = F_{N+N-1}^2 - F_{n-1}^2 \tag{31}$$
$$m_{2,n} = F_{n+N}^2 - F_n^2 \tag{32}$$
$$r_{1,n} = F_{n+N} - F_{n+N-1} \tag{33}$$
$$r_{2,n} = F_n - F_{n-1} \tag{34}$$

and consider F_{n-1}^2 being aperiodic. Then, when $r_{1,n}F_{n+N} - r_{2,n}F_n = 0$, $G_{n+N} - G_n = m_{1,n} + \frac{1}{4}(\beta - \alpha^2)(m_{1,n} + m_{2,n} - 2m_{2,n})$ is not always equal to zero. Also, when $2(r_{1,n}F_{n+N} - r_{2,n}F_n) = l_n \neq 0$, $G_{n+N} - G_n = m_{1,n} + \frac{1}{4}(\beta - \alpha^2)(m_{1,n} + m_{2,n} - 2m_{2,n} + l_n)$ is not always equal to zero. Therefore, the above assumption about the periodicity of G_n is not valid. In other words, G_n is aperiodic.

Theorem 3: For Type 3 systems, the Kalman gain k_n in (23) converges to zero when the systems are periodic, and k_n oscillates aperiodically when the systems are chaotic.

Proof: Since G_n is aperiodic, Type 3 systems can be arranged to Type 2 systems, and the result follows from Lemma 1.

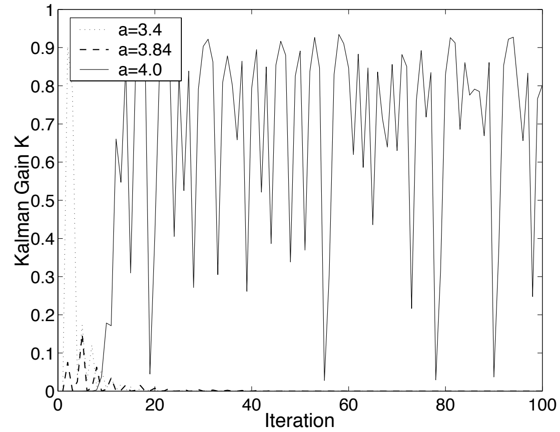

Fig. 1. Convergence characteristic of the Kalman gain for tent map with different values of a corresponding to periodic signals (a = 0.3 and a = 0.4) and chaotic signals (a = 0.8 and a = 0.9). Dotted lines denote the theoretical steady states.

III. Computer Simulations

To illustrate the theoretical results developed in the foregoing section, we consider three systems, i.e., the tent map and two logistic maps.

A. Type 1

We consider the following class of tent maps:

$$x_{n+1} = a(1 - |2x_n - 1|)\,, \tag{35}$$

where $a \in [0, 1]$ and $x_n \in [0, 1]$. The global Lyapunov exponent and local exponent of the system are identical, and equal to $\ln 2a$. The map is chaotic for $a > \frac{1}{2}$, and is periodic otherwise. The UKF algorithm for this map can be realized by first using (31) to get the Jacobian matrix, i.e.,

$$F_n = \begin{cases} 2a, & x_n < \frac{1}{2} \\ -2a, & x_n > \frac{1}{2}. \end{cases} \tag{36}$$

Then, we have $F_n^2 = 4a^2$, which is independent of x_n. According to Theorem 1, the Kalman gain converges to

$$\lim_{n\to\infty} k_n = \begin{cases} 0, & a < \frac{1}{2} \\ \frac{4a^2-1}{4a^2}, & a > \frac{1}{2} \end{cases}. \tag{37}$$

Figure 1 shows the typical convergence behavior of the Kalman gain for this map, in which the dotted lines denote the steady states given by (37). We can see from Fig. 1 that the magnitude of the fixed point increases as the Lyapunov exponent gets larger when the system is chaotic, as given in Theorem 1.

B. Type 2

We consider the logistic map given by

$$x_{n+1} = ax_n(1 - x_n)\,, \tag{38}$$

where $a \in [0, 4]$ and $x_n \in [0, 1]$. The Jacabian matrix of the map is $F_n = a - 2ax_n$, and the Hession matrx is $-2a$. Thus, this map is a Type 2 system.

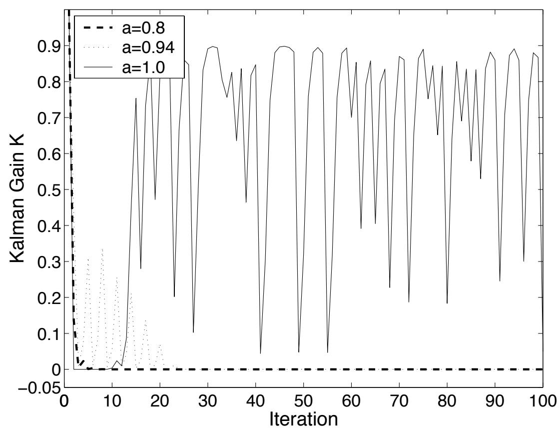

Fig. 2. Convergence characteristic of the Kalman gain of the logistic map (Type 2). Dotted, dashed, and solid lines describes the k_n's convergence behavior for the system with parameter $a = 3.4$ (periodic), $a = 3.84$ (periodic), and $a = 4.0$ (chaotic), respectively.

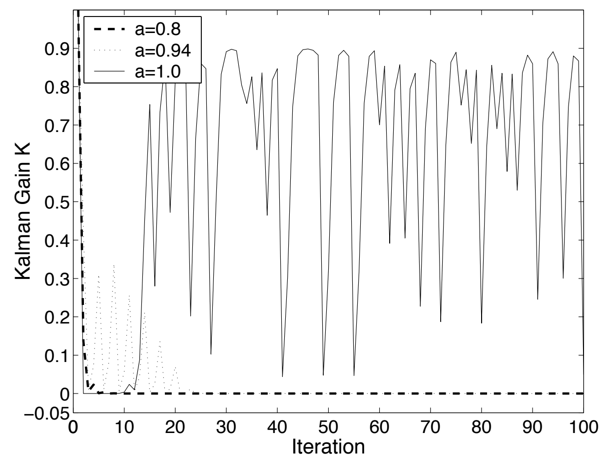

Fig. 3. Convergence characteristic of the Kalman gain for the logistic map (Type 3). Dotted, dashed and solid lines describes k_n's behavior for the system with parameter $a = 0.8$ (periodic), $a = 0.94$ (periodic) and $a = 1.0$ (chaotic), respectively.

Figure 2 shows the convergence behavior of k_n, in which the dotted, dashed, and solid lines describe the k_n's convergence behavior for the system with parameter $a = 3.4$, $a = 3.84$ and $a = 4.0$, respectively. The former two are periodic, while the latter gives a chaotic time series. When the time series is periodic, k_n converges to the fixed zero state. However, when the time series is chaotic, k_n exhibits an aperiodic behavior, as described in Theorem 2.

C. Type 3

We consider another form of logistic map, which is defined as

$$x_{n+1} = a \sin(\pi x_n) \,, \tag{39}$$

where $a \in [0, 1.45]$ and $x_n \in [-1.5, 1.5]$. The Jacobian matrix of the map is $F_n = a\pi \cos(\pi x_n)$, and the Hession matrix is $H_n = -a\pi^2 \sin(\pi x_n)$, which are dependent on x_n. This map is therefore a Type 3 system.

Figure 3 shows the convergence behavior of k_n, in which the dotted, dashed, and solid lines correspond to the results of $a = 0.8$, $a = 0.94$ and $a = 1.0$, respectively. The former two are periodic, while the latter gives a chaotic time series. When the time series is periodic, k_n converges to the fixed zero state. However, when the time series is chaotic, k_n exhibits an aperiodic behavior, as described in Theorem 3.

IV. Conclusion

The problem for filtering a chaotic system from noisy observation by using the unscented Kalman filter algorithm has been investigated in this paper. It has been proven that when a nonlinear system is chaotic, the Kalman gain of the UKF does not converge to zero, but either converges to a fixed point with magnitude larger than zero or oscillates aperiodically. The dynamical behavior of the error covariance matrix can be readily found since it is linearly related to the Kalman gain.

Acknowledgment

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