design example for wireless communication application shows that this architecture uses less hardware area and consumes less power.

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Analysis of the Convergence and Divergence of a Constrained Anti-Hebbian Learning Algorithm

Clifford Sze-Tsan Choy and Wan-Chi Siu

Abstract—In this paper, we analyze the effect of initial conditions on a constrained anti-Hebbian learning algorithm suggested by Gao, Ahmad, and Swamy. Although their approach has a minimum memory requirement with simple computation, we demonstrate through a simple example that divergence is always possible when the initial state satisfies suitable condition. We point out that in analyzing their learning rule, a constrained differential equation has to be considered instead of the unconstrained one they have studied in their original paper. Furthermore, we analyze this constrained differential equation and prove that 1) it diverges under similar conditions and 2) there is only one stable equilibrium whose domain of attraction we have identified. Accordingly, we suggest a re-initialization approach for the learning rule, which leads to convergence and yet preserves the simplicity of the original approach with a slight increase in computation.

Index Terms—Constrained anti-Hebbian learning algorithm, convergence and divergence analysis, stochastic approximation, total least square fit.

I. INTRODUCTION

Many engineering problems, like system identifications and parameter estimations, can be formulated as fitting a line to a set of data subject to errors or noises. Let us denote a set of data $D = \{\xi_i = \begin{bmatrix} \mathbf{x}_i \\ d_i \end{bmatrix}, i = \{1, \dots, m\}\}$, where $\mathbf{x}_i$ is a column vector in the n -dimensional space and d_i is a scalar. In fitting a line $L: d = \mathbf{a}^T \mathbf{x} + b$ to these data (where $\mathbf{X}^T$ is the transpose of the matrix $\mathbf{X}$), a conventional criterion is to fit in the least-squares (LS) sense. The LS criterion assumes that errors are confined to the "observation" $\{d_i\}$, hence it minimizes the error function $E_{ls}(\mathbf{a}, b)$ where

$$E_{ls}(\mathbf{a}, b) = \sum_{i=1}^m |d_i - \mathbf{a}^T \mathbf{x}_i - b|^2. \quad (1)$$

When errors are also present in the inputs $\{\mathbf{x}_i\}$, then the total least-squares (TLS) criterion [1] gives a better fit. Fitting according to the TLS criterion is to find $\{\mathbf{a}_{tls}, b_{tls}\}$ which minimizes $E_{tls}(\mathbf{a}, b)$, where

$$E_{tls}(\mathbf{a}, b) = \sum_{i=1}^m \frac{|d_i - \mathbf{a}^T \mathbf{x}_i - b|^2}{\sqrt{1 + |\mathbf{a}^T \mathbf{x}_i|^2}}. \quad (2)$$

Geometrically, the solution in TLS sense is one which minimizes the sum of squared distances from each datum to the nearest point on the estimated line. Recently, the TLS criterion has been applied to solve a number of problems, including frequency estimation [2], curve and surface fitting [3], adaptive filtering [4], [5], electromagnetic emitter localization [6], and deconvolution [7].

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In order to fit a line $L: d = \mathbf{a}^T \mathbf{x} + b$ to the set of data D defined earlier, we define the data matrix $\mathbf{E}$ from D

$$\mathbf{E} = \begin{bmatrix} \xi_1 - \mathbf{e} & \xi_2 - \mathbf{e} & \cdots & \xi_m - \mathbf{e} \end{bmatrix}^T \quad (3)$$

where

$$\mathbf{e} = \frac{1}{m} \sum_{i=1}^m \xi_i.$$

By applying the singular value decomposition (SVD) on the data matrix, we have

$$\mathbf{E} = \mathbf{U} \mathbf{W} \mathbf{V}^T \quad (4)$$

where $\mathbf{U}$ and $\mathbf{V} = [\nu_1 \ \nu_2 \ \cdots \ \nu_{n+1}]$ are unitary matrices such that $\mathbf{U}^T \mathbf{U} = \mathbf{I}_m$ and $\mathbf{V}^T \mathbf{V} = \mathbf{I}_{n+1}$, where $\mathbf{I}_k$ is an identity matrix with dimension k , and $\mathbf{W}$ is an $m \times (n+1)$ matrix in the form $\begin{bmatrix} \mathbf{Z} \\ 0 \end{bmatrix}$ where $\mathbf{Z} = \text{diag}(\sigma_1, \dots, \sigma_{n+1})$ is a diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > \sigma_{n+1}$. Let us denote $c_{n+1, n+1}$ as the value of the $(n+1)$ th dimension of ν_{n+1} . When $c_{n+1, n+1} \neq 0$, the TLS solution $\mathbf{a}_{\text{tls}}$ is given by [1], [3]

$$\begin{bmatrix} \mathbf{a}_{\text{tls}} \\ -1 \end{bmatrix} = -\frac{\nu_{n+1}}{c_{n+1, n+1}} \quad b_{\text{tls}} = -\mathbf{a}_{\text{tls}}^T \mathbf{e}. \quad (5)$$

The eigenvectors of the covariance matrix $\mathbf{C} = (1/m) \mathbf{E}^T \mathbf{E}$ of D are represented by $\{\nu_1, \dots, \nu_{n+1}\}$, where ν_{n+1} is the eigenvector having the smallest eigenvalue $\lambda_{n+1} = \sigma_{n+1}^2$.

Indeed, to simplify analysis, it is sufficient to consider a set of data D which has zero mean (i.e., $\mathbf{e} = \mathbf{0}$), and the TLS solution is obtained by determining the eigenvector corresponding to the smallest eigenvalue of the correlation matrix $\mathbf{R} = (1/m) \mathbf{E}^T \mathbf{E}$ of D . In the following, we denote the set of eigenvectors of $\mathbf{R}$ as $\{\nu_1, \dots, \nu_{n+1}\}$ and the corresponding eigenvalues as $\{\lambda_1, \dots, \lambda_{n+1}\}$, such that λ_{n+1} is the smallest eigenvalue with multiplicity one. Also, we assume that $c_{n+1, n+1} \neq 0$.

Although SVD can be used to find solution in the TLS sense, it is too cumbersome to be applied, especially in real-time applications. Recently, anti-Hebbian learning algorithms were proposed as on-line approaches to find the TLS solution [3], [5], [8]. In [3], two constrained Hebbian learning algorithms in [9] and [10] were modified into two constrained anti-Hebbian learning algorithms by using a negative learning rate. Let us denote $\Psi(t)$ as the weight vector ($n+1$ dimensions) and $\xi(t)$ as an input vector from a stationary distribution, then the two constrained anti-Hebbian learning algorithms are

$$\Psi(t+1) = \Psi(t) - \mu(t) \eta(t) [\xi(t) - \eta(t) \Psi(t)] \quad (6a)$$

$$\Psi(t+1) = \Psi(t) - \mu(t) \eta(t) \left[\xi(t) - \frac{\eta(t) \Psi(t)}{\Psi(t)^T \Psi(t)} \right] \quad (6b)$$

where

$$\eta(t) = \Psi(t)^T \xi(t).$$

By assuming the statistical independence between $\Psi(t)$ and $\xi(t)$, some assumptions on $\xi(t)$ according to [10] [including $\mathbf{R} = \varepsilon(\xi(t) \xi(t)^T)$ where $\varepsilon(\cdot)$ denotes statistical averaging operation] and $\mu(t)$ is allowed to decrease to zero in an appropriate way according to stochastic approximation literatures (e.g., [11], [12],

proportional to $1/t$), they stated that the two learning rules could, respectively, be approximated by the following differential equations:

$$\frac{d\hat{\Psi}(\tau)}{d\tau} = -\mathbf{R} \hat{\Psi}(\tau) + [\hat{\Psi}(\tau)^T \mathbf{R} \hat{\Psi}(\tau)] \hat{\Psi}(\tau) \quad (7a)$$

and

$$\frac{d\hat{\Psi}(\tau)}{d\tau} = -\mathbf{R} \hat{\Psi}(\tau) + \frac{\hat{\Psi}(\tau)^T \mathbf{R} \hat{\Psi}(\tau)}{\hat{\Psi}(\tau)^T \hat{\Psi}(\tau)} \hat{\Psi}(\tau). \quad (7b)$$

The approximation of (7a) to (6a) [(7b) to (6b)] is in the sense that the asymptotic paths of (7a) and (6a) [(7b) and (6b)] are close with a large probability and eventually the solution $\Psi(t)$ of (6a) [(6b)] tends, with probability one, to the uniformly asymptotically stable solution of (7a) [(7b)] (see [10] for detail). Xu *et al.* applied the work of Oja and Karhunen [10] to prove that (6b) converges to ν_{n+1} (the *minor component*) of the correlation matrix $\mathbf{R}$ with probability one. On the other hand, the convergence analysis of (7a) indicates potential divergence of $\hat{\Psi}(\tau)$, which we will review in Section III-A. Hence, even though (6b) has higher computational complexity than (6a) [the Euclidean norm of $\Psi(t)$ has to be evaluated and an extra division operation is required to explicitly normalize $\Psi(t)$ in the former], (6b) is more appealing theoretically.

As pointed out in Gao *et al.* [5], $n+1$ variables have to be updated in (6a) and (6b), and an extra division operation has to be performed according to (5) in obtaining the TLS solution $\mathbf{a}_{\text{tls}}$. In order to reduce complexity, they proposed a constrained anti-Hebbian learning algorithm as follows:

$$\Psi(t+1) = \Psi(t) - \mu(t) \eta(t) (\xi(t) + \xi_{n+1}(\tau) \Psi(t)) \quad (8)$$

where

$$\eta(t) = \Psi(t)^T \xi(t).$$

They showed that by (8), $\Psi_{n+1}(\tau) = -1$ for any time t , where $\Psi_{n+1}(\tau)$ is the $n+1$ th dimension of $\Psi(t)$. Owing to this fact, only n variables have to be updated. Indeed, (8) has the simplicity of the well-known least-mean-squares (LMS) algorithm [13]. By assuming statistical independence between $\Psi(t)$ and $\xi(t)$, and that $\mathbf{R} = \varepsilon(\xi(t) \xi(t)^T)$, they showed that the averaged equation of (8) becomes

$$\varepsilon(\Psi(t+1) | \Psi(t)) = \Psi(t) - \mu(t) [\mathbf{R} \Psi(t) + \Psi(t)^T \mathbf{r} \Psi(t)] \quad (9a)$$

where

$$\mathbf{r} = \sum_{i=1}^{n+1} \lambda_i c_{i, n+1} \nu_i. \quad (9b)$$

By using the same assumptions as in the work of [3] stated in the previous paragraph, (8) is approximated by the following differential equation:

$$\frac{d\hat{\Psi}(\tau)}{d\tau} = -\mathbf{R} \hat{\Psi}(\tau) - \hat{\Psi}(\tau)^T \mathbf{r} \hat{\Psi}(\tau). \quad (10)$$

In addition, they proved the following theorem ([5], Theorem 1), which describes the asymptotic property of (10).

Theorem 1: Let $\mathbf{R}$ be a semipositive definite matrix with the smallest eigenvalue of multiplicity of one, ν_{n+1} be the corresponding normalized eigenvector, and $c_{n+1, n+1} \neq 0$ be the last component of ν_{n+1} . Then if $\hat{\Psi}(0)^T \nu_{n+1} \neq 0$, $\hat{\Psi}(\tau)$ tends to be in the direction of ν_{n+1} asymptotically as $\tau \rightarrow \infty$. $\square$

Then, they concluded that $\Psi(t)$ would converge to the TLS solution $\mathbf{a}_{\text{tls}}$, without any division operation as in [3].

In the work of Luo *et al.* [8], they proposed the following anti-Hebbian learning rule:

$$\Psi(t+1) = \Psi(t) - \mu(t) \left(\Psi(t)^T \Psi(t) \eta(t) \xi(t) - \eta(t)^2 \Psi(t) \right) \quad (11)$$

where

$$\eta(t) = \Psi(t)^T \xi(t)$$

Under similar assumptions and arguments in [3] and [5], (11) is approximated by the following differential equation:

$$\frac{d\hat{\Psi}(\tau)}{d\tau} = -\hat{\Psi}(\tau)^T \hat{\Psi}(\tau) \mathbf{R} \hat{\Psi}(\tau) + \hat{\Psi}(\tau)^T \mathbf{R} \hat{\Psi}(\tau) \hat{\Psi}(\tau) \quad (12)$$

where $\mathbf{R} = \varepsilon(\xi(t)\xi(t)^T)$ as usual. They proved that

$$\lim_{\tau \rightarrow \infty} \hat{\Psi}(\tau) = \pm \hat{\Psi}(0)^T \hat{\Psi}(0) \nu_{n+1} \quad (13)$$

under the assumption that $\hat{\Psi}(0)^T \nu_{n+1} \neq 0$ and λ_{n+1} is of multiplicity one.

Comparing the three anti-Hebbian learning rules, an obvious advantage of Gao *et al.*'s approach is that it is the simplest computationally and does not require the extra division operation in determining the TLS fit. The objective of this paper is to analyze the convergence and divergence of (8) due to initial conditions. Although Gao *et al.* claimed that (8) would converge directly from any initial random position to the position that is aligned in the direction of the eigenvector corresponding to the smallest eigenvalue of the correlation matrix $\mathbf{R}$, in next section, we are going to give a geometric interpretation of their learning rule and demonstrate with a simple example that this is not true (with the weight vector diverges to infinity) using certain initial conditions for any positive learning rate $\mu(t)$. While (8) has divergence property, this seems to contradict with the convergence property of the differential equation in (10) as guaranteed by Theorem 1. We note that this seemingly contradictory observation is due to the reason that (10) is actually *not* approximated by (8). In Section III, we will illustrate this point by referring to their proof. Indeed, we will argue that (8) is approximated by the constrained differential equation—(10) *subject to the constraint* $\hat{\Psi}_{n+1}(\tau) = -1$. We suggest a new theorem which defines the domain of attraction of this constrained differential equation and conditions under which it diverges. We will present experimental results to indicate the validity of these criteria. Comparison of the convergence condition of (10) subject to the constraint with that of (7a) indicates that the former converges to the correct result under more reasonable conditions than the latter. In Section V, we suggest an approach such that when (8) starts to diverge, the weight vector is “projected” onto the domain of attraction of the constrained differential equation according to our new theorem. This preserves the simplicity of (8). Finally, we conclude our work.

II. DIVERGENCE OF THE LEARNING RULE

With Ψ as the weight vector and ξ as a sampled data vector, we can decompose Ψ relative to ξ into two components

$$\Psi = \Psi_{\perp} + \Psi_{\parallel} \quad (14)$$

where $\Psi_{\perp}$ is the component perpendicular to ξ and $\Psi_{\parallel}$ is the component parallel to ξ . Simple geometry leads us to the following:

$$\Psi_{\parallel} = \frac{\Psi^T \xi}{|\xi|^2} \xi \quad (15a)$$

$$\Psi_{\perp} = \Psi - \Psi_{\parallel}. \quad (15b)$$

Let us consider the following updating equation:

$$\Delta \Psi = \alpha \Psi_{\parallel} + \beta \Psi_{\perp}. \quad (16)$$

Equating (8) and (16), we have

$$\alpha = -\mu(|\xi|^2 + \eta \xi_{n+1}) \quad (17a)$$

$$\beta = -\mu \eta \xi_{n+1} \quad (17b)$$

where η is defined in (8).

Suppose it is true that

$$|\xi|^2 + \eta \xi_{n+1} < 0. \quad (18)$$

Since $|\xi|^2$ is nonnegative for any data vector, it must be the case that

$$-\eta \xi_{n+1} > |\xi|^2 \geq 0. \quad (19)$$

Hence, as long as $\mu > 0$, it is true that $\beta > 0$ and $\alpha > 0$. Referring to (8), the value of β is just the gain in the self-feedback term (i.e., $-\mu \eta \xi_{n+1} \Psi$). Note that for any $\mu > 0$, the value of β must be positive, according to (19). Suppose it is true that the learning rate $\mu(>0)$ decreases sufficiently slow as in stochastic approximation methods (e.g., [11], [12], [14]), and that $|\xi|$ is bounded. Then, if it is highly probable that (18) is satisfied during learning, it is likely that $|\Psi|$ will diverge to infinity, independent of the choice of μ .

We are going to illustrate a simple example when $n = 1$, in which (18) holds persistently during learning. In this example, any data vector $\xi_i = \begin{bmatrix} x_i \\ d_i \end{bmatrix}$ satisfies the relation $d_i = a \cdot x_i$ (i.e., data vectors are not corrupted by any noise). Fig. 1 illustrates the case when $a > 0$. Note that any data vector lies on the line $L: \begin{bmatrix} a \\ -1 \end{bmatrix}^T \xi = 0$, and that the vector $\begin{bmatrix} a \\ -1 \end{bmatrix}$ is parallel to ν_2 (i.e., the eigenvector with smallest eigenvalue). With $\Psi = \begin{bmatrix} b \\ -1 \end{bmatrix}$, let us consider the case when $b < -(1/a)$ as shown. Then, we have

$$\begin{aligned} |\xi|^2 + \eta \xi_{n+1} &= \left\| \begin{bmatrix} x \\ ax \end{bmatrix} \right\|^2 + \begin{bmatrix} b \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ ax \end{bmatrix} ax \\ &= x^2 \cdot a \cdot \left(\frac{1}{a} + b \right) \\ &< 0 \text{ since } b < -\frac{1}{a} \text{ by construction.} \end{aligned} \quad (20)$$

This means that α and β in (16) are both positive according to our analysis in the previous paragraph. Furthermore, the perpendicular component of Ψ relative to ξ is given by

$$\Psi_{\perp} = \begin{bmatrix} b \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} b \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ ax \end{bmatrix}}{\left\| \begin{bmatrix} x \\ ax \end{bmatrix} \right\|^2} \begin{bmatrix} x \\ ax \end{bmatrix} = -\frac{1+ab}{1+a^2} \begin{bmatrix} -a \\ 1 \end{bmatrix}. \quad (21)$$

Fig. 1. A simple example illustrating the divergence of (8) due to inappropriate initial conditions. (See text for details.)

Since $-(1+ab)/(1+a^2) > 0$ when $b < -(1/a)$, $\Psi_\perp$ is pointing at the direction of $\begin{bmatrix} -a \\ 1 \end{bmatrix}$, which is also pointing toward the negative direction of x -axis. Together with the fact that $\alpha > 0$, we know that Δb must be negative. Hence, whenever we start with $b < -(1/a)$, this will stay true for any positive learning rate μ . As a result, b will diverge to negative infinity with sufficiently long time, for any positive μ . The same argument is applicable to the case when $a < 0$ and $b > -(1/a)$, such that $b \rightarrow +\infty$ as $t \rightarrow \infty$ for any positive μ .

This example gives us an insight into why (8) diverges. The learning rule as expressed in (16) may be interpreted as an orthogonalization scheme, such that the component of Ψ parallel to a data vector ξ is reduced and its perpendicular component is enhanced, subject to the constraint that $\Psi_{n+1} = -1$. Due to this constraint, the only solution of Ψ is given by $-(\nu_{n+1}/c_{n+1, n+1})$. However, the vector $\nu_{n+1}/c_{n+1, n+1}$ is also parallel to the eigenvector ν_{n+1} . Since the perpendicular component to be enhanced depends on where Ψ is in $\mathcal{R}^{n+1}$, it can point to either direction [i.e., $-(\nu_{n+1}/c_{n+1, n+1})$ or $\nu_{n+1}/c_{n+1, n+1}$]. Due to the constraint $\Psi_{n+1} = -1$, enhancing the component in the direction of $-(\nu_{n+1}/c_{n+1, n+1})$ is the only way to have a correct solution. In our example, it is the direction $\begin{bmatrix} -a \\ 1 \end{bmatrix}$ to be enhanced, which means that there will be no solution for Ψ such that $\Psi^T \begin{bmatrix} a \\ -1 \end{bmatrix} = 0$. Hence, the divergence problem is attributed to an initialization in the wrong subspace.

III. CONVERGENCE AND DIVERGENCE OF THE CONSTRAINED DIFFERENTIAL EQUATION

Let us consider the recursive algorithm

$$\Psi(t+1) = \Psi(t) + \mu(t)q(\Psi(t), \xi(t)) \quad (22)$$

where $\xi(t)$ is bounded, is sampled from a stationary distribution, and is independent of $\Psi(t)$. According to the theory of stochastic approximation (e.g., [12], [14]), under suitable conditions, this recursive algorithm is approximated by the associated differential equation

$$\frac{d\hat{\Psi}(\tau)}{d\tau} = f(\hat{\Psi}(\tau)) \quad (23)$$

where $f(\Psi) = \varepsilon(q(\Psi, \xi))$. In fact, as noted by Ljung [12], (22) can be studied and analyzed in terms of the differential equation (23) under suitable conditions. Indeed, when the learning rule is

sufficiently small, (23) is a good approximation to the averaged equation of the recursive equation in (22).

Hence, in order to understand the convergence and divergence of (8), we have to analyze its associated differential equation. While the divergence of the learning rule in (8) has been demonstrated in the previous section, this seems to contradict with Theorem 1 which guarantees the convergence of (10). The reason is that Theorem 1 considers only an *unconstrained* differential equation, while (8) is constrained [with $\hat{\Psi}_{n+1}(\tau) = -1$]. Hence, in analyzing the convergence and divergence of (8), it is more appropriate to consider the constrained differential equation

$$\frac{d\hat{\Psi}(\tau)}{d\tau} = -\mathbf{R}\hat{\Psi}(\tau) - \hat{\Psi}(\tau)^T \mathbf{r} \hat{\Psi}(\tau)$$

subject to

$$\hat{\Psi}_{n+1}(\tau) = -1. \quad (24)$$

In Section III-A, we will summarize the approaches in proving the convergence of (7a) [3] and the proof of Theorem 1 in [5]. The approach of Xu *et al.* is included since similar concept has been applied in the proof of Gao *et al.* Also, the condition for convergence is relatively strict according to their proof, for which we will give comments later. We will point out that Theorem 1 is not applicable in analyzing the convergence of (24). Then, in Section III-B, a new theorem is presented, indicating the sufficient conditions for convergence and divergence of (24). After this, we will comment on our proof.

A. Review on Previous Approaches

1) *The Approach of Xu et al. [3] and Luo et al. [8]:* The approach to proving the convergence of (7a) [3] and that in proving the convergence of (12) are similar to that in [9] and [10]. For our current study, we will summarize the main points in the work of Xu *et al.* [3] as follows.

The weight vector $\hat{\Psi}(\tau)$ was decomposed relative to the orthonormal basis $\{\nu_1, \dots, \nu_{n+1}\}$ (i.e., the eigenvectors of the correlation matrix $\mathbf{R}$) such that

$$\hat{\Psi}(\tau) = \sum_{i=1}^{n+1} v_i(\tau) \nu_i. \quad (25)$$

They then obtained

$$\begin{aligned} \frac{dv_i(\tau)}{d\tau} &= -\lambda_i v_i(\tau) + \left(\hat{\Psi}(\tau)^T \mathbf{R} \hat{\Psi}(\tau) \right) v_i(\tau) \\ &\quad \forall i \in \{1, \dots, n+1\}. \end{aligned} \quad (26)$$

By requiring $v_{n+1}(0) \neq 0$, they noted that $v_{n+1}(\tau) \neq 0$ for all τ . Therefore, they defined the ratio $\zeta_i(\tau) = (v_i(\tau)/v_{n+1}(\tau))$, $\forall i \in \{1, \dots, n\}$, and obtained the following relation:

$$\frac{d\zeta_i(\tau)}{d\tau} = (\lambda_{n+1} - \lambda_i) \zeta_i(\tau), \quad \forall i \in \{1, \dots, n\}. \quad (27)$$

Since $\lambda_{n+1} > \lambda_i$, $\forall i \in \{1, \dots, n\}$, $\zeta_i(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. Due to this reason, they stated that the following should be true asymptotically:

$$\hat{\Psi}(\tau) = v_{n+1}(\tau) \nu_{n+1} \quad (28)$$

and that (26) for $v_{n+1}(\tau)$ became

$$\frac{dv_{n+1}(\tau)}{d\tau} = -\lambda_{n+1}(v_{n+1}(\tau)^3 - v_{n+1}(\tau)). \quad (29)$$

Based on (29), they concluded the following.

- 1) If $\lambda_{n+1} = 0$, then $v_{n+1}(\tau) = v_{n+1}(0)$ for all τ , and $\hat{\Psi}(\tau)$ tended to $v_{n+1}(0)\nu_{n+1} = \hat{\Psi}(0)^T \nu_{n+1}$;
- 2) If $\lambda_{n+1} > 0$, then $|v_{n+1}(\tau)|$ tended to either zero (if $|v_{n+1}(\tau)| < 1$) or infinity (if $|v_{n+1}(\tau)| > 1$), and the point $v_{n+1}(\tau) = 1$ was an unstable fixed point.

It is obvious from their conclusion that (7a) converges only when $\lambda_{n+1} = 0$, and $\hat{\Psi}(\tau)$ may diverge or reduce to zero when this is not satisfied.

2) *The Approach of Gao et al.:* In proving Theorem 1, they applied the same concept as above. The projection of $\hat{\Psi}(\tau)$ onto the i th eigenvector was given by

$$\frac{dv_i(\tau)}{d\tau} = -\lambda_i v_i(\tau) - \left(\sum_{k=1}^{n+1} \lambda_k c_{k,n+1} v_k(\tau) \right) v_i(\tau) \quad \forall i \in \{1, \dots, n+1\}. \quad (30)$$

With the same requirement that $v_{n+1}(0) \neq 0$, the same differential equations for the ratio $\zeta_i(\tau) = (v_i(\tau)/v_{n+1}(\tau))$ as in (27) were obtained. Using the same argument as above that $\hat{\Psi}(\tau) = v_{n+1}(\tau)\nu_{n+1}$ asymptotically, they reduced (30) for $v_{n+1}(\tau)$ into

$$\frac{dv_{n+1}(\tau)}{d\tau} = -\lambda_{n+1} v_{n+1}(\tau)(1 + c_{n+1,n+1} v_{n+1}(\tau)). \quad (31)$$

According to (31), they concluded the following.

- 1) For $\lambda_{n+1} > 0$, (31) could have an asymptotically stable equilibrium at zero.
- 2) For $\lambda_{n+1} = 0$, $v_{n+1}(\tau) = v_{n+1}(0)$ for all τ .

It is clear that the condition for convergence (to correct solution) is as restrictive as that of Xu *et al.* in that $\hat{\Psi}(\tau)$ will converge to the direction of ν_{n+1} when $\lambda_{n+1} = 0$, while $|\hat{\Psi}(\tau)| \rightarrow 0$ otherwise. Even when the algorithm converges to a nonzero vector, $\hat{\Psi}(\tau) = (\hat{\Psi}(0)^T \nu_{n+1}) \nu_{n+1}$ as $\tau \rightarrow \infty$ according to their proof, which may not satisfy the constraint $\hat{\Psi}_{n+1}(\tau) = -1, \forall \tau$.

Let us illustrate the problem when their proof is applied to analyze the convergence of (24). With $v_i(\tau)$ as defined in (25), the constraint $\hat{\Psi}_{n+1}(\tau) = -1$ becomes

$$\sum_{i=1}^{n+1} c_{i,n+1} v_i(\tau) = -1. \quad (32)$$

Obviously, $v_{n+1}(\tau)$ is dependent on $\{v_1(\tau), \dots, v_n(\tau)\}$. If one assumes that $(v_i(\tau)/v_{n+1}(\tau)) \rightarrow 0$ as $\tau \rightarrow \infty$, we have $c_{n+1,n+1} v_{n+1} = -1$ according to (32), without any need to consider the convergence of the differential equation in (31). However, in so doing, one is *assuming* the convergence of (24) to the valid solution, which should not be considered as a valid proof. This indicates that Theorem 1 is not applicable to this constrained case.

B. The Proposed Theorem

Let us propose the following new theorem.

Theorem 2: Let $\mathbf{R}$ be a semipositive definite $(n+1) \times (n+1)$ matrix, whose normalized eigenvectors are $\{\nu_1, \dots, \nu_{n+1}\}$ and

the corresponding eigenvalues are $\{\lambda_1, \dots, \lambda_{n+1}\}$, such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > \lambda_{n+1} \geq 0$ (i.e., λ_{n+1} has multiplicity of one). Also, the last component of ν_{n+1} is nonzero, i.e., $c_{n+1,n+1} \neq 0$, and $\hat{\Psi}(0)^T \nu_{n+1} \neq 0$. Then, as $\tau \rightarrow \infty$, (24) will

- 1) converge, with $\hat{\Psi}(\tau) \rightarrow -(\nu_{n+1}/c_{n+1,n+1})$, when

$$c_{i,n+1} \hat{\Psi}(0)^T \nu_i \leq 0$$

and

$$c_{n+1,n+1} \hat{\Psi}(0)^T \nu_{n+1} < 0, \quad \forall i \neq n+1 \quad (33)$$

- 2) diverge, when $c_{n+1,n+1} \hat{\Psi}(0)^T \nu_{n+1} > 0$.

Indeed, D_c forms the invariant set of (24), where

$$D_c = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : \forall i \neq n+1, c_{i,n+1} \mathbf{x}^T \nu_i \leq 0 \text{ and } c_{n+1,n+1} \mathbf{x}^T \nu_{n+1} < 0 \right\}. \quad (34)$$

(Note: An invariant set of a differential equation is a set such that each trajectory of this differential equation which starts within this set remains there for all subsequent times.) $\square$

Proof of Theorem 2: By decomposing $\hat{\Psi}(\tau)$ relative to the eigenvectors $\{\nu_1, \dots, \nu_{n+1}\}$ as in (25), the same set of equations in (30) can be arrived, which is our starting point. In this proof, we make use of the fact that due to the constraint in (32), one of the variables in $\{v_1(\tau), \dots, v_{n+1}(\tau)\}$ can be expressed as a combination of the other n variables. Hence, one of the differential equations in (30) is redundant, and we can consider a system of n differential equations without constraint. To simplify the notation in the following discussions, we define $e_i(\tau) = c_{i,n+1} v_i(\tau), \forall i \in \{1, \dots, n+1\}$.

Suppose it is the case that $\forall i \in \{1, \dots, n\}, \forall \tau, e_i(\tau) = 0$. Due to the constraint in (32), we have $e_{n+1}(\tau) = -1$, which means that $v_{n+1}(\tau) = -(1/c_{n+1,n+1}), \forall \tau$ (since we have required $c_{n+1,n+1} \neq 0$). Since the value of $v_{n+1}(\tau)$ is known, we only need to consider the n differential equations for $v_1(\tau)$ to $v_n(\tau)$, subject to no constraint. For this special case, we have, according to (30)

$$\frac{dv_i(\tau)}{d\tau} = -(\lambda_i - \lambda_{n+1}) v_i(\tau), \quad \forall i \in \{1, \dots, n\} \quad (35)$$

whose solution is $v_i(\tau) = v_i(0) \exp\{-(\lambda_i - \lambda_{n+1})\tau\}$. Since $\lambda_i > \lambda_{n+1} \forall i \in \{1, \dots, n\}$, we know that $v_i(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ for any value of $v_i(0)$. Together with the fact that $v_{n+1}(\tau) = -(1/c_{n+1,n+1})$ for all τ , we have $\hat{\Psi}(\tau) \rightarrow -(\nu_{n+1}/c_{n+1,n+1})$ as $\tau \rightarrow \infty$. This proves a special case of convergence.

Otherwise, we can always choose an index $q \in \{1, \dots, n\}$, such that $e_q(0) \neq 0$, which necessitates $c_{q,n+1} \neq 0$. Rewriting (32), we have

$$v_q(\tau) = \frac{1}{c_{q,n+1}} \left\{ -1 - \sum_{\substack{i=1 \\ i \neq q}}^{n+1} c_{i,n+1} v_i(\tau) \right\}. \quad (36)$$

Substituting (36) into (30), we arrive at the following n unconstrained differential equations:

$$\begin{aligned} \frac{dv_i(\tau)}{d\tau} = & -(\lambda_i - \lambda_q)v_i(\tau) \\ & - \left(\sum_{\substack{k=1 \\ k \neq q}}^{n+1} (\lambda_k - \lambda_q)c_{k,n+1}v_k(\tau) \right) v_i(\tau), \\ & \forall i \in \{1, \dots, n+1\} \setminus \{q\} \end{aligned} \quad (37)$$

such that the variable $v_q(\tau)$ is removed due to the constraint $\hat{\Psi}_{n+1}(\tau) = -1$.

Similar to the approach in [3], [5], and [10], we define $\vartheta_i(\tau) = (v_i(\tau)/v_{n+1}(\tau))$ for $i \in \{1, \dots, n\} \setminus \{q\}$ and obtain the following differential equations:

$$\begin{aligned} \frac{d\vartheta_i(\tau)}{d\tau} = & -((\lambda_i - \lambda_q) - (\lambda_{n+1} - \lambda_q))\vartheta_i(\tau), \\ & \forall i \in \{1, \dots, n\} \setminus \{q\}. \end{aligned} \quad (38)$$

The solutions of these $n-1$ differential equations are

$$\begin{aligned} v_i(\tau) = & \frac{v_i(0)}{v_{n+1}(0)} \exp\{-(\lambda_i - \lambda_{n+1})\tau\} v_{n+1}(\tau), \\ & \forall i \in \{1, \dots, n\} \setminus \{q\}. \end{aligned} \quad (39)$$

Since we have required that λ_{n+1} has a multiplicity of one, it is obvious that $\lambda_i - \lambda_{n+1} > 0$ for $i \in \{1, \dots, n\} \setminus \{q\}$. Hence, as long as $v_{n+1}(\tau)$ is bounded, $v_i(\tau) \rightarrow 0$ as $t \rightarrow \infty$ for all $i \in \{1, \dots, n\} \setminus \{q\}$.

By substituting (39) into the differential equation for $v_{n+1}(\tau)$ in (37), and then multiplying both sides by $c_{n+1,n+1}$ (since we have required $c_{n+1,n+1} \neq 0$), we have

$$\begin{aligned} \frac{de_{n+1}(\tau)}{d\tau} = & -(\lambda_{n+1} - \lambda_q)e_{n+1}(\tau) \\ & - \left(\sum_{\substack{k=1 \\ k \neq q}}^{n+1} (\lambda_k - \lambda_q) \frac{e_k(0)}{e_{n+1}(0)} \exp\{-(\lambda_k - \lambda_{n+1})\tau\} \right) \\ & \cdot e_{n+1}(\tau)^2. \end{aligned} \quad (40)$$

Note that we have required $e_{n+1}(0) = c_{n+1,n+1}v_{n+1}(0) = \hat{\Psi}(0)^T \nu_{n+1} \neq 0$. The differential equation for $e_{n+1}(\tau)$ is indeed a Bernoulli equation, and its closed-form solution (e.g., according to [15, p. 2]) is

$$\begin{aligned} \frac{1}{e_{n+1}(\tau)} = & C \exp\{-(\lambda_q - \lambda_{n+1})\tau\} - \sum_{\substack{k=1 \\ k \neq q}}^{n+1} \frac{e_k(0)}{e_{n+1}(0)} \\ & \cdot \exp\{-(\lambda_k - \lambda_{n+1})\tau\} \end{aligned} \quad (41)$$

where

$$C = \frac{1}{e_{n+1}(0)} + \sum_{\substack{k=1 \\ k \neq q}}^{n+1} \frac{e_k(0)}{e_{n+1}(0)}.$$

Due to the constraint in (32), we have $\sum_{k=1}^{n+1} e_k(\tau) = -1$, and the value of C in (41) becomes

$$C = \frac{1}{e_{n+1}(0)} \left(1 + \sum_{\substack{k=1 \\ k \neq q}}^{n+1} e_k(0) \right) = -\frac{e_q(0)}{e_{n+1}(0)}. \quad (42)$$

Hence, we have

$$\frac{1}{e_{n+1}(\tau)} = -\sum_{k=1}^{n+1} \frac{e_k(0)}{e_{n+1}(0)} \exp\{-(\lambda_k - \lambda_{n+1})\tau\}. \quad (43)$$

Based on the constraint in (32), together with the solutions of $v_i(\tau)$, $\forall i \in \{1, \dots, n\} \setminus \{q\}$ in (39) and the solution of $v_{n+1}(\tau)$ in (43), the expression for $e_q(\tau)$ is

$$\frac{1}{e_q(\tau)} = -\sum_{k=1}^{n+1} \frac{e_k(0)}{e_q(0)} \exp\{-(\lambda_k - \lambda_q)\tau\}. \quad (44)$$

Note that q is selected such that $e_q(0) \neq 0$.

Let us consider the asymptotic solution of $\hat{\Psi}(\tau)$ as $t \rightarrow \infty$. In this case, $e_{n+1}(\tau) \rightarrow -1$ and $v_i(\tau) \rightarrow 0$ for $i \in \{1, \dots, n\}$. This corresponds to the desired solution $\hat{\Psi}(\tau) \rightarrow -(\nu_{n+1}/c_{n+1,n+1})$ as $\tau \rightarrow \infty$. Note that this is the only stable solution of (24).

However, the trajectory of $\hat{\Psi}(\tau)$ will diverge under some conditions. Precisely, if there exists τ'' such that $1/e_q(\tau'') = 0$ or $1/e_{n+1}(\tau'') = 0$, the quantity $|\hat{\Psi}(\tau')|$ goes to infinity. Let us consider the value of $e_{n+1}(\tau)$. According to (43), the continuity of $1/e_{n+1}(\tau)$ in τ is obvious. Since $e_{n+1}(\tau) \rightarrow -1$ as $\tau \rightarrow \infty$ (which is the only asymptotic solution), if for some time τ' , $e_{n+1}(\tau') > 0$, the trajectory of $1/e_{n+1}(\tau)$ will intersect the τ -axis at some finite time $\tau'' > \tau'$. This means that $e_{n+1}(\tau'') = \pm\infty$ and the trajectory of $\hat{\Psi}(\tau)$ diverges. Hence, a sufficient condition for divergence of (24) is that $e_{n+1}(0) > 0$.

Suppose it is true that $\forall i \in \{1, \dots, n\}$, $e_i(0) \leq 0$ and $e_{n+1}(0) < 0$. Under this condition, $1/e_{n+1}(\tau) < 0$, $\forall \tau$ [according to (43)]. Similarly, it is true that $1/e_q(\tau) < 0$, $\forall \tau$ [according to (44)]. Since these two quantities do not change their signs, the values of $e_q(\tau)$ and $e_{n+1}(\tau)$ are bounded for all τ . Furthermore, due to the boundedness of $e_{n+1}(\tau)$, the values of $v_i(\tau)$ for $i \in \{1, \dots, n\} \setminus \{q\}$ in (39) are bounded as well. Hence, under this condition, the trajectory of $\hat{\Psi}(\tau)$ is bounded and converges to $-(\nu_{n+1}/c_{n+1,n+1})$ as $\tau \rightarrow \infty$. This proves the sufficient condition for convergence of (24).

Finally, let us assume again that $\forall i \in \{1, \dots, n\}$, $e_i(0) \leq 0$ and $e_{n+1}(0) < 0$. Multiplying both sides of (39) by $c_{i,n+1}$ for each $i \in \{1, \dots, n\} \setminus \{q\}$, we have

$$\begin{aligned} e_i(\tau) = & \frac{e_i(0)}{e_{n+1}(0)} \exp\{-(\lambda_i - \lambda_{n+1})\tau\} e_{n+1}(\tau), \\ & \forall i \in \{1, \dots, n\} \setminus \{q\}. \end{aligned} \quad (45)$$

Since the exponential function is positive, and $e_i(0)/e_{n+1}(0) \geq 0$, the fact that $e_{n+1}(\tau) < 0$ implies that $e_i(\tau) \leq 0$ for all $\tau > 0$. In addition, according to the last paragraph, $e_q(\tau) < 0$ for all $\tau > 0$ under this condition. This proves that the trajectory stays within the set D_c for all $\tau > 0$ when $\hat{\Psi}(0) \in D_c$, where D_c is as defined in (34). Hence, D_c is the invariant set of (24).

This completes our proof. $\square$

It is obvious from the proof that the only stable solution of the constrained differential equation in (10) is the desired one. However, its trajectory diverges at some finite time under some initial conditions. According to our theorem, the constrained differential equation diverges when $c_{n+1, n+1} \hat{\Psi}(0)^T \nu_{n+1} > 0$. This agrees with the divergence condition in our simple example in Section II. Note that Theorem 1 cannot predict the divergence since the theorem only considers an unconstrained differential equation in (10). This shows the inadequacy of Theorem 1 in studying the learning rule in (8).

According to the above proof, convergence to the desired solution is possible under the condition that $\Psi(0) \in D_c$, where D_c is defined in (34), and that the smallest eigenvalue has the multiplicity of one. This shows that the constrained differential equation in (10) is theoretically better than (7a) or the unconstrained one in (10), since the latter two approaches converge to the desired solution under relatively restrictive condition of $\lambda_{n+1} = 0$. Actually, this corresponds to the case when the variance along the minor component is zero, or equivalently, when the TLS solution perfectly fits the data, which is very rare in real life applications.

IV. EXPERIMENTS

A number of experiments have been done to test the validity of our analysis. To illustrate our idea, let us estimate an “unknown” finite impulse response (FIR) system, which has been used by Gao *et al.* as an example in [5, Section V.B]. The “unknown” FIR system has the form

$$d(t) = \mathbf{w}^* T \mathbf{u}(t) + n_o(t) \quad (46)$$

where

$$\begin{aligned} \mathbf{w}^* &= [-0.3, -0.9, 0.8, -0.7, 0.6]^T \\ \mathbf{u}(t) &= [u(t), u(t-1), u(t-2), u(t-3), u(t-4)]^T \\ \mathbf{x}(t) &= [u(t) + n_i(t), u(t-1) + n_i(t-1), u(t-2) \\ &\quad + n_i(t-2), u(t-3) + n_i(t-3), u(t-4) \\ &\quad + n_i(t-4)]^T \end{aligned}$$

with input noise $n_i(t)$ and output noise $n_o(t)$, both are zero-mean white Gaussian noise with a variance of 0.25, and $u(t)$ is a random signal uniformly distributed over $[-0.5, 0.5]$ as the stimulating signal. The vector to be learnt ξ in (8) is given by $[\mathbf{x}(t)]$. We generated a training set D with 1000 samples and performed SVD on the correlation matrix $\mathbf{R} = (1/1000) \sum_{\xi \in D} \xi \xi^T$ to obtain the eigenvector corresponding to the smallest eigenvalue as $\nu_{n+1} = [-0.128424, -0.505533, 0.436667, -0.384574, 0.327158, -0.531353]^T$. For the learning rule, we used an initial weight vector $\Psi(0) = [x, 1, 1, 1, 1, -1]^T$, where $x = 0, 1, 2, 3, 4, 5, 6$. Note that when $x > 3.154$, we have $c_{n+1, n+1} \Psi(0)^T \nu_{n+1} > 0$, which satisfies the sufficient condition for divergence in Theorem 2. The learning rate was fixed at 0.005 throughout our experiments.

For each value of x , ten independent trials were performed, each using a different random seed for the random generator in sampling input vectors. Fig. 2 shows two plots of the quantity $y(t) = \sum_{i=1}^{10} c_{n+1, n+1} \Psi_i(t)^T \nu_{n+1}$ against the number of iterations t , where $\Psi_i(t)$ corresponds to the trajectory of the i th independent trial. Note that in case of convergence, the value of $y(t)$ tends to -1 . Fig. 2(a) shows the plots when $x = 0, 1, 2, 3$. According to our simulations, the learning rule in (8) diverges when $x = 3$, and converges when $x = 0, 1$, and 2. However, divergence when

Fig. 2. (a) When $x = 0, 1, 2, 3, 4$ in which convergence is observed. (b) When $x = 5, 6$ in which divergence is observed. The case when $x = 4$ is included as a reference.

$x = 3$ does not mean that Theorem 2 is wrong. On one hand, it has to be recalled $x > 3.154$ is a sufficient condition for divergence, but not a necessary condition, according to Theorem 2. On the other hand, $x > 3.154$ is the divergence condition for (24), which is an approximation to (8). Fig. 2(b) shows the plots when $x = 0, 4, 5, 6$. The plot of $x = 0$ is included as a reference. It shows that when $x = 4, 5$, and 6, the learning rule diverges.

V. DISCUSSIONS

Although the learning rule of Gao *et al.* in (8) is potentially divergent, if it is initialized in the appropriate region (i.e., within D_c in Theorem 2) convergence is possible. In fact, as long as it converges, it is a better approach as compared to (6a), since as guaranteed by Theorem 2, its convergence to the minor component of the correlation matrix of the input data is possible under a less restrictive condition, as we have discussed in the last paragraph of Section III. Also, it requires a lesser amount of computation than that of (6b), in which the latter requires explicit normalization. Furthermore, the TLS solution is readily available without further division, as contrary to the two anti-Hebbian learning rules.

Note that a simple initial condition for the learning rule in (8) is that $\Psi(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, where $\mathbf{0}$ denotes an n -dimensional zero vector. With this condition, we have

$$c_{i, n+1} \Psi(0)^T \nu_i = -c_{i, n+1}^2 \leq 0 \quad \forall i \in \{1, \dots, n+1\} \quad (47)$$

in which the equality holds iff $c_{i,n+1} = 0$. Obviously, this initialization is well within the invariant set D_c in (34), hence convergence is possible. Even with this initialization, convergence cannot be guaranteed, unless the learning rate is sufficiently small such that the constrained differential equation in (24) approximates sufficiently close to the trajectory of the learning rule. Under this situation, the trajectory of (8) stays infinitely often inside the invariant set D_c , so that we can apply theory in stochastic approximation (e.g., [12, Theorem 1]) to guarantee that the learning rule converges to the minor component with probability one under suitable conditions.

There may be a situation when the learning rule has to be applied with arbitrary initialization, e.g., in tracking a slowly time-varying source. A possible solution is to check for divergence of the learning rule, such that if it diverges, the weight vector $\Psi(t)$ is “projected” [12], [14] onto the invariant set D_c in (34), i.e., to reinitialize it from a new point in D_c where the learning rule is started again. An obvious choice of this “projected” point is $\Psi(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, which is guaranteed to be within D_c in any case as shown in the preceding paragraph.

Detecting for divergence is possible if we know the dynamic range of each dimension of the desired solution $-(\nu_{n+1}/c_{n+1,n+1})$. This is possible, for example, in estimating parameters in adaptive FIR filtering, since coefficients in the filters are normally bounded. Then, a limit on $|\Psi(t)|$ can be defined, and divergence is assumed when this limit is exceeded. Another possible way is to monitor the value of $|\mu(t)\Psi(t)^T\xi(t)\xi_{n+1}(t)|$ during learning, which is the learning rate of the self-feedback term of $\Psi(t)$ in (8). Suppose if we can bound the value of $|\xi(t)|$ and $|\xi_{n+1}(t)|$ for any t (note that an almost sure bound will do), then we can derive an upper bound for $|\Psi(t)^T\xi(t)\xi_{n+1}(t)|$. Accordingly, the maximum value of $\mu(t)$ during learning can be determined, since convergence is possible when $|\mu(t)\Psi(t)^T\xi(t)\xi_{n+1}(t)| < 1$. However, when divergence is due to initial conditions, the value of $|\mu(t)\Psi(t)^T\xi(t)\xi_{n+1}(t)|$ will be increasing at a very fast rate, which is due to the corresponding rapid increase in $|\Psi(t)|$. Hence, a consequence of the divergence is a rapid increase in this quantity. Since we have already bounded the value of $\mu(t)$, whenever it is the case that $|\mu(t)\Psi(t)^T\xi(t)\xi_{n+1}(t)| \gg 1$, we can assure that the algorithm is diverging. It is obvious that $\mu(t)\Psi(t)^T\xi(t)\xi_{n+1}(t)$ is readily available during learning, since it is the learning rate and we have to calculate it anyway. Hence, checking the boundedness of $|\mu(t)\Psi(t)^T\xi(t)\xi_{n+1}(t)|$ is simpler than computing the quantity $|\Psi(t)|$.

Let us consider the above “projection” approach such that the learning rate $\mu(t)$ is decreased sufficiently slow as in stochastic approximation literatures (a popular requirement in existing stochastic approximation literatures [11], [12], [14] is that

$$\lim_{t \rightarrow \infty} \mu(t) = 0, \quad \sum_{t=1}^{\infty} \mu(t) = \infty \quad \text{and} \quad \sum_{t=1}^{\infty} \mu(t)^2 < \infty \quad (48)$$

such that the sequence is decreased sufficiently slow and yet convergence is possible, e.g., the sequence $\mu(t) = 1/t$). Suppose it is true that $\xi(t)$ is independent of $\Psi(t)$, $\xi(t)$ is bounded for all t and the expectation of $\xi(t)\xi(t)^T$ exists which is denoted by $\mathbf{R}$. Then, in the following, we outline the proof in guaranteeing the convergence of our “projection” approach with probability one to the minor component of $\mathbf{R}$ under the condition that the smallest eigenvalue has a multiplicity of one. As usual, the dimensionality of vectors is $n+1$, the eigenvalues of $\mathbf{R}$ are denoted by $\{\lambda_1, \dots, \lambda_{n+1}\}$, and its eigenvectors are denoted by $\{\nu_1, \dots, \nu_{n+1}\}$. Our “projection” approach is equivalent to one in which reinitialization is performed whenever the weight vector $\Psi(t)$ is outside the set $D_1 = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| < C, \text{ where } C \text{ is the bound on } \Psi(t) \text{ for nondivergent, as}$

defined in previous paragraph. Due to the projection, the trajectory stays infinitely often in time in D_1 . If it is the case that the trajectory stays infinitely often in time in D_c , we can apply [12, Theorem 1] to prove the convergence of this “projection” approach to the minor component with probability one. In order to prove the above, let us suppose the contrary that the trajectory of $\Psi(t)$ stays infinitely often within the region $D_1 \setminus D_c$. It is obvious that $\Psi(t)$ is bounded. Since $\xi(t)$ is bounded as well, the fact that the learning rate decreases to zero implies that the value of $\Psi(t)$ converges to a stable solution within the region $D_1 \setminus D_c$. Applying [12, Theorem 2], $\Psi(t)$ can only converge to the stable solution of the constrained differential equation in (24), which we have shown in the proof of Theorem 2 to be $-(\nu_{n+1}/c_{n+1,n+1})$. However, this stable solution is *not* inside the set $D_1 \setminus D_c$, which contradicts our assumption that the trajectory of $\Psi(t)$ stays infinitely often in time in this region. Hence, this proves the convergence of our “projection” approach.

VI. CONCLUSIONS

In this paper, we have made an analysis on the convergence and divergence of the anti-Hebbian learning rule from Gao *et al.* in [5] due to initial conditions only. We have presented a geometrical interpretation of their learning rule and demonstrated with a simple example that divergence is always possible when the weight vector is initialized in the wrong subspace. Although they suggested a proof on the convergence of a differential equation in analyzing the convergence of their learning rule, we note that their proof concerns an unconstrained differential equation. Since the learning rule is constrained, we claim that it is more appropriate to consider a constrained differential equation instead, and show that their proof is not applicable in analyzing this constrained differential equation. Indeed, we have proved a theorem which gives the sufficient conditions for convergence and divergence of the constrained differential equation. According to our theorem, convergence to the desired solution [i.e., $-(\nu_{n+1}/c_{n+1,n+1})$] is possible when λ_{n+1} has a multiplicity of one. In comparison to the differential equation considered by Gao *et al.* and the associated differential equation of one of the anti-Hebbian learning rule from Xu *et al.* [3], this condition is less restrictive, since these two approaches require $\lambda_{n+1} = 0$ for converging to the desired solution. The condition for divergence agrees with the condition we have observed in our simple example. Experimental results have been presented to verify the correctness of the divergence condition.

Based on our new theorem, we located the invariant set of the constrained differential equation, and proved that the initial condition $\Psi(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ (where $\Psi(t)$ is an $(n+1)$ -dimensional vector and $\mathbf{0}$ is an n -dimensional zero vector) can lead to convergence under appropriate conditions. Accordingly, we have also suggested a “projection” approach in which the weight vector $\Psi(t)$ is “projected” or reinitialized to $\Psi(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ whenever divergence is observed. This solution preserves the simplicity of the learning rule from Gao *et al.*

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Integrated Circuit Signal Measurements Using an Undersampling Approach

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Abstract—Integrated circuit (IC) manufacturing processes have been successful in introducing complex high-speed analog and mixed-signal devices. Testing these devices is becoming increasingly difficult. The paper presents a novel method of analyzing analog IC's using periodic input stimuli and wide-band undersampling. In its simplest form, the testing procedure can be implemented in a design by adding an analog switch to sample the response signal at a particular node under test and a buffer to bring the sampled values off-chip. Using a sequential undersampling algorithm to control the switch allows high-frequency signals to be mixed down in frequency and driven off-chip using a low bandwidth buffer. By placing the sampling circuit on-chip, the high-frequency buffering problems associated with a similar system using digital sampling scopes or mixed-signal IC testers can be avoided. The utility of the procedure has been illustrated by measuring the frequency response, slew rate, and transient response characteristics for a unity gain 1.2- μm CMOS opamp.

Index Terms—High speed, IC measurements, mixed signal, undersampling.

I. INTRODUCTION

Advances in integrated circuit (IC) processing technologies have enabled the development of complex high-speed analog and mixed-signal devices [1]–[3]. Process advances have resulted in a steady increase in circuit density and speed. Although mixed-signal technologies have become a popular solution to many design problems, current testing technologies are inadequate for accessing high-frequency analog signals on-chip [2], [3].

A common testing procedure used to determine the signal level at a particular node is to buffer the signal off-chip [4]. Some of the problems with this approach include: 1) intrinsic parasitic capacitance of the buffer input stage and metal interconnect will load the node under test; 2) one output pin is required for each test node unless some form of multiplexing is used (multiplexing introduces additional parasitics and limits the frequency response); and 3) the large load on the output pin can severely limit the high-frequency response.

Another common approach used to test analog IC's is circuit probing [5]–[7]. Contact probing techniques have problems similar to those encountered with off-chip buffering. The node under test is loaded by the parasitic characteristics of the probe testing apparatus and the on-chip probe pad [5], [6]. Noncontact probing technologies can significantly reduce loading [7], however, they are generally not suitable for production tests where high pin counts and high throughput are required.

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