Very fast discrete Fourier transform, using number theoretic transform

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Abstract: It is shown that number theoretic transforms (NTT) can be used to compute discrete Fourier transform (DFT) very efficiently. By noting some simple properties of number theory and the DFT, the total number of real multiplications for a length- P DFT is reduced to $(P-1)$. This requires less than one real multiplication per point. For a proper choice of transform length and NTT, the number of shift adds per point is approximately the same as the number of additions required for FFT algorithms.

1 Introduction

Direct computation of length- N discrete Fourier transform [1] (DFT) requires N^2 multiplications. The number of multiplications reduces to $\frac{1}{2}N \log_2 N$ if the fast Fourier transform algorithm [2] (FFT) is used. Winograd [3] showed that the minimum number of multiplications required to compute the circular convolution of two length- N sequences is $2N - K$, where K is the number of divisors of N including 1 and N . Agarwal and Cooley [4], Winograd [5] and Kolba and Parks [6] made use of Rader's theorem [7] on DFT with prime transform length to construct their algorithms for the computation of DFT. Compared to conventional FFT method, the Winograd Fourier transform algorithms reduce the number of multiplications by a factor of two to three, with a slightly large number of additions. Reed and Truong [8] proposed a technique for the computation of discrete Fourier transforms, based on Winograd's method in combination with Mersenne prime number-theoretic transforms. This hybrid algorithm requires fewer multiplications than either the standard FFT or Winograd's more conventional algorithm. However, a very large number of additions are required and the number of multiplications per point is still relatively large (about 1.33 to 3.49 multiplications per point). Recently, Nussbaumer [9] first defined a type of polynomial transform over the field of polynomials which could be used to compute 2-dimensional convolutions efficiently. This leads to the development of fast algorithms for the computation of multidimensional DFT [10, 11]. The number of multiplications in this case reduces to just two or three per point for sequences even longer than 1000 points.

In this paper, it is shown that the number of multiplications can be reduced further by using number theoretic transforms [12–15] to evaluate the DFT, and this forms a very efficient method of calculating the DFT.

2 Theory

Let the residue of the number g^n modulo P be written as $\langle g^n \rangle_P$, where g is a primitive root that generates all nonzero elements inside the field modulo P . Consider now an N -point discrete Fourier transform

$$Y(k) = \sum_{n=0}^{N-1} x(n) W_0^{nk} \quad (1)$$

where $k = 0, 1, \dots, N-1$ and

$$W_0 = e^{-j(2\pi/N)}$$

If N is a prime number P , then eqn. 1 can be reordered [7] in the following form:

$$Y(0) = \sum_{n=0}^{P-1} x(n) \quad (2)$$

and

$$Y(\langle g^k \rangle_P) = x(0) + \sum_{n=1}^{P-1} x(\langle g^{-n} \rangle_P) W_0^{\langle g^{k-n} \rangle_P} \quad \text{for } k = 1, 2, \dots, P-1 \quad (3)$$

We can write eqn. 3 as

$$Y(\langle g^k \rangle_P) = x(0) + X(\langle g^k \rangle_P) \quad (4)$$

where

$$X(\langle g^k \rangle_P) = \sum_{n=1}^{P-1} x(\langle g^{-n} \rangle_P) W_0^{\langle g^{k-n} \rangle_P} \quad \text{for } k = 1, 2, \dots, P-1 \quad (5)$$

Eqn. 5 represents a backward circular convolution of length $(P-1)$. That is,

$$[x(g^{-1}), x(g^{-2}), \dots, x(g^{-P+1})] \circledast [W_0^{g^0}, W_0^{g^1}, \dots, W_0^{g^{P-2}}] \quad (6)$$

where $\circledast$ means circular convolution and the subscripts and indices are modulo P .

Let us now define

$$X_k = X(\langle g^{k+1} \rangle_P) \quad (7)$$

$$W_n = W_0^{\langle g^n \rangle_P} \quad (8)$$

$$x_n = x(\langle g^{-(n+1)} \rangle_P) \quad (9)$$

$$\text{for } k = 0, 1, \dots, P-2; n = 0, 1, \dots, P-2$$

Hence, eqns. 5 and 6 become

$$X_k = \sum_{n=0}^{P-2} x_n W_{k-n} \quad \text{for } k = 0, 1, \dots, P-2 \quad (10)$$

and

$$(x_0, x_1, \dots, x_{P-2}) \circledast (W_0, W_1, \dots, W_{P-2}) \quad (11)$$

The number theoretic transform can now be applied to find the cyclic convolution sum of these two sequences. Thus we can write

$$X'_m = \left\langle \sum_{n=0}^{P-2} x_n \alpha^{mn} \right\rangle_M \quad (12)$$

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and

$$W'_m = \left\langle \sum_{n=0}^{P-2} W_n \alpha^{mn} \right\rangle_M \quad \text{for } m = 0, 1, \dots, P-2 \quad (13)$$

where α = a root of unity of order $(P-1)$ and M = base for modulo arithmetic.

The results can then be obtained by the inverse transform of the products, $X'_m W'_m$. That is,

$$X_k = \left\langle \frac{1}{P-1} \sum_{m=0}^{P-2} X'_m W'_m \alpha^{-mk} \right\rangle_M \quad \text{for } k = 0, 1, \dots, P-2 \quad (14)$$

Recall that all W_m s are complex numbers; hence apparently the total number of multiplications for a real sequence of length P (to find all $X'_m W'_m$) for this method is $2(P-1)$. However, the sequence $(W_0^0, W_0^1, \dots, W_0^{P-2})$ can actually be written as [5]

$$\{W_0^0, W_0^1, \dots, W_0^{[(P-1)/2]-1}, W_0^{g^0*}, W_0^{g^1*}, \dots, W_0^{g^{[(P-1)/2]-1}*}\}$$

where $*$ denotes complex conjugates.

Therefore, the sequence $(W_0, W_1, \dots, W_{P-2})$ can be written as

$$\{W_0, W_1, \dots, W_{[(P-1)/2]-1}, W_0^*, W_1^*, \dots, W_{[(P-1)/2]-1}^*\}$$

Notice also that $\text{Real}(W_n) = \text{Real}(W_n^*)$ and $\text{Imag}(W_n) = -\text{Imag}(W_n^*)$. Hence,

$$\text{Real}\{W_{[n+(P-1)/2]}\} = \text{Real}(W_n) \quad (15)$$

$$\text{Imag}\{W_{[n+(P-1)/2]}\} = -\text{Imag}(W_n) \quad (16)$$

$$\text{for } n = 0, 1, \dots, \left(\frac{P-1}{2} - 1\right)$$

In view of these relationships, eqn. 13 can be written as

$$\begin{aligned} W'_m &= \left\langle \sum_{n=0}^{[(P-1)/2]-1} W_n \alpha^{mn} + \sum_{n=[(P-1)/2]}^{P-2} W_n \alpha^{mn} \right\rangle_M \\ &= \left\langle \sum_{n=0}^{[(P-1)/2]-1} W_n \alpha^{mn} \right. \\ &\quad \left. + \sum_{n=0}^{[(P-1)/2]-1} W_{[n+(P-1)/2]} \alpha^{[n+(P-1)/2]m} \right\rangle_M \end{aligned}$$

and, since $\alpha^{[(P-1)/2]} = -1$, we can write

$$\begin{aligned} W'_m &= \left\langle \sum_{n=0}^{[(P-1)/2]-1} W_n \alpha^{mn} \right. \\ &\quad \left. + (-1)^m \sum_{n=0}^{[(P-1)/2]-1} W_{[n+(P-1)/2]} \alpha^{mn} \right\rangle_M \quad (17) \end{aligned}$$

On combining eqns. 15–17, it is clear that

$$\begin{aligned} \text{Real}(W'_m) &= \text{Real} \left(\left\langle 2 \sum_{n=0}^{[(P-1)/2]-1} W_n \alpha^{mn} \right\rangle_M \right) \\ &\quad \text{for } m = \text{even} \\ &= 0 \quad \text{for } m = \text{odd} \quad (18) \end{aligned}$$

$$\begin{aligned} \text{Imag}(W'_m) &= 0 \quad \text{for } m = \text{even} \\ &= \text{Imag} \left(\left\langle 2 \sum_{n=0}^{[(P-1)/2]-1} W_n \alpha^{mn} \right\rangle_M \right) \\ &\quad \text{for } m = \text{odd} \quad (19) \end{aligned}$$

Eqns. 18 and 19 are very important in practical implementations, and this is not primarily because the number of shift (α multiplication if α is a simple combination of power of two) adds reduces by a factor of two for the calculation of W'_m , since all W'_m should be precalculated for hardware implementation; but because, however, the total number of real multiplications forming $X'_m W'_m$ reduces from $(2P-2)$ to $(P-1)$. This gives less than one [actually $1 - (1/P)$] multiplication per point for the DFT of a real sequence of length P .

The number of additions required in this technique is evaluated below. The shift adds required by computing W'_m are not counted, since these quantities can be precalculated and stored in ROM for hardware implementation or stored in program for software implementation. The total number of shift adds required for transforming $(x_0, x_1, \dots, x_{P-2})$ to $(X'_0, X'_1, \dots, X'_{P-2})$ is $(P-1)(P-2)$. However, if we choose $(P-1)$ to be highly composite, an FFT-type algorithm can be applied to effect the transformation. In particular, if $(P-1)$ is a power of two, the number of shift adds is approximately equal to $(P-1) \log_2(P-1)$. Since the results are complex, two inverse transformations, one real and one imaginary, are required. Owing to the symmetry property of the DFT, only the first half of the length- $(P-1)$ inverse transform is necessary to compute. The other half of the inverse transform can be obtained by taking the conjugate of the first half of the inverse transform. Furthermore, both real and imaginary parts of the sequences $(X'_m W'_m, m = 0, 1, \dots, P-2)$ are alternately zero, a length- $(P-1)$ inverse transformation can be formed by two length- $[(P-1)/2]$ inverse transforms. Hence, the number of shift-adds for the inverse transformation is

$$2 \left(\frac{P-1}{2} \right) \left(\frac{P-1}{2} - 1 \right) = (P-1) \left(\frac{P-1}{2} - 1 \right)$$

in general, or is

$$2 \left(\frac{P-1}{2} \right) \log_2 \left(\frac{P-1}{2} \right) = (P-1) [\log_2(P-1) - 1]$$

if $(P-1)$ is a power of two. The total number of shift adds for $(P-1)$ being a power of two is $(P-1)[2 \log_2(P-1) - 1]$. Therefore, the overall number of shift adds including the additions of $x(0)$ and the additions for $Y(0)$ becomes

$$\begin{aligned} (P-1)[2 \log_2(P-1) - 1 + 2] \\ = (P-1)[2 \log_2(P-1) + 1] \end{aligned}$$

This figure of shift adds is approximately equal to the number of real additions for FFT. Hence, the number of operations is significantly less than the number of operations reported in Reference 8.

3 Example

To illustrate the idea, let us consider the DFT of the sequence $[x(0), x(1), x(2), x(3), x(4)]$, i.e. $N = P = 5$. In this case, 2 is a primitive root which is used to generate elements inside the field modulo 5. Hence the mapping for $[x(n)]$ in eqns. 3 and 5 is given by

$$\begin{aligned} (x_n: n = 1, 2, 3, 4) &= [x(\langle g^{-n} \rangle_M): n = 1, 2, 3, 4] \\ &= [x(3), x(4), x(2), x(1)] \end{aligned}$$

and

$$(W_0^g: n = 0, 1, 2, 3) = (W_0^1, W_0^2, W_0^4, W_0^3)$$

Hence, eqn. 6 becomes:

$$[x(3), x(4), x(2), x(1)] \odot (W_0^1, W_0^2, W_0^4, W_0^3) \\ [x(3), x(4), x(2), x(1)] \odot (W_0^1, W_0^2, W_0^{1*}, W_0^{2*}) \quad (20)$$

This convolution sum can be computed by NTT. Now let us use Fermat number transform (FNT) to make the calculation. Let $M = F_4 = 2^{16} + 1$, $\alpha = 2^8$ and, of course, $N = 4$. Hence,

$$\begin{bmatrix} X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{bmatrix} = \left\langle \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2^8 & -1 & -2^8 \\ 1 & -1 & 1 & -1 \\ 1 & -2^8 & -1 & 2^8 \end{bmatrix} \begin{bmatrix} x(3) \\ x(4) \\ x(2) \\ x(1) \end{bmatrix} \right\rangle_M \quad (21)$$

$$\begin{bmatrix} W'_0 \\ W'_1 \\ W'_2 \\ W'_3 \end{bmatrix} = \left\langle \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2^8 & -1 & -2^8 \\ 1 & -1 & 1 & -1 \\ 1 & -2^8 & -1 & 2^8 \end{bmatrix} \begin{bmatrix} W_0^1 \\ W_0^2 \\ W_0^{1*} \\ W_0^{2*} \end{bmatrix} \right\rangle_M \quad (22)$$

where $W_0 = e^{-j(2\pi/5)}$.

In order to use modulo arithmetic, the W_0 terms have to be normalised to integer values. Multiplying these terms by 90 and rounding off the results to integers, we obtain:

$$\begin{bmatrix} W'_0 \\ W'_1 \\ W'_2 \\ W'_3 \end{bmatrix} = \begin{bmatrix} -90 + j0 \\ 0 + j38229 \\ 202 + j0 \\ 0 + j26964 \end{bmatrix}$$

This expression may be compared with eqns. 18 and 19 for agreement. Hence, for the computation of $X'_m W'_m$, $m = 0, 1, 2, 3$, a total number of four real multiplications is sufficient. This is also the total number of multiplications required for a 5-point DFT. The total number of real shift adds required is $4(2 \log_2 4 + 1) = 20$.

As we have seen, the length for the NTT is $(P - 1)$, where $(P - 1)$ is always an even number. Fermat number transforms [14], pseudo Mersenne transforms [16], pseudo Fermat transforms [17], or any efficient transform with even number of transform length, are suitable for the computation. However, for some very promising NTTs, the transform lengths may not be long enough or may not match this requirement. For example, an excellent choice of P is 257, which is prime, and $N (= P - 1)$ is highly composite, and it might be possible to use NTT to effect the convolution. The longest transform length (with $\sqrt{2}$ as the generator) for FNT with modulo base F_6 is 256. Hence F_6 is a possible choice for the implementation. If one wishes to use a shorter word-length, F_5 say, to make the implementation the major problem is that the maximum transform length for the FNT with $M = F_5 = 2^{32} - 1$ is 128 for $\alpha = \sqrt{2}$. However, this problem may be resolved by using multidimensional techniques for convolutions.* Since P is a prime number, it is also possible to combine Winograd's short DFTs to carry out the computation of long DFTs using multidimensional formulations [18]. The major disadvantages of the method using NTT to calculate DFT are that special arithmetic (modulo arithmetic) and normally relatively large word lengths may have to be used for the major part of the calculation—a fact common to all number theoretic transforms.

* The 1-dimensional 256-point cyclic convolution can be converted into a 2×128 -point 2-dimensional convolution form. The 128-point cyclic convolution can be found by using FNT, whereas the other dimension is actually a length-2 linear convolution which can be computed by the Lagrange interpolation formula. If $(P - 1)$ is equal to the product of two mutually prime integers, the 1-dimensional cyclic convolution can be converted into a proper 2-dimensional cyclic convolution which can be computed by two number theoretic transforms or by a combination of the Winograd's convolution algorithm and the NTT.

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